Geophys. J. R. astr. Soc. (1972) **29**, 329–366.

The Theory of Stress Wave Radiation from Explosions in Prestressed Media

Charles B. Archambeau

(Received 1972 January 24)

Summary

Stress wave radiation from underground explosions has been observed to contain an anomalous shear wave contribution which is most likely of tectonic origin. In this paper the theoretical radiation field to be expected from an explosion in a prestressed medium is given under the assumption that no secondary low symmetry faulting on a large scale occurs and that the total tectonic component of the field is due to stress relaxation around the roughly spherical fracture zone created by the explosive shock wave. Evidence for the occurrence of this simple kind of tectonic source is considered, and it is concluded that this model is appropriate in many, if not most, instances involving underground explosions. Expressions for the spectrum of the radiation field and its spatial radiation pattern are given in terms of multipole expansions for the components of the rotation potential and the dilatation potential. Several possible rupture formation models are treated. All models show that the tectonic radiation is of simple quadrupole form, as has been observed. The energy radiated due to stress relaxation is considered in detail, and it is also shown that, in terms of the energy released, a dislocation source can be used as an equivalent for the stress relaxation effects.

The theoretical energy partition between compressional and shear waves for the tectonic field is in the ratio of (approximately) 1 to 10, so that tectonic stress release does not affect the direct compressional body wave particularly, but gives rise to totally anomalous *SH* polarized waves (e.g. Love waves) and affects Rayleigh type surface waves significantly, as is also observed. The theory can be applied to obtain estimates of source dimensions and the orientation and magnitude of the initial prestress field in the region of the explosion. In addition, application of this particular form of the general tectonic source theory to deep earthquakes and volcanic earthquakes also appears to be reasonable in view of the probable high symmetry of the failure or phase transition regions for such events.

1. Introduction

Observations of the elastic radiation field from large underground explosions have led numerous investigators (e.g. Press & Archambeau 1962; Toksöz, Harkrider & Ben-Menahem 1965; Brune & Pomeroy 1963; Aki 1964; Archambeau & Sammis

1970) to conclude that the anomalous part of the radiation field is due to some form of induced tectonic stress release. The anomalous tectonic energy is thought to arise from either a relatively large-scale failure of the medium, most likely along a pre-existing fault zone in the vicinity of the explosion (Brune & Pomeroy 1963; Aki 1964; Aki *et al.* 1969), or else from stress relaxation in the vicinity of the roughly spherically symmetric crushed zone created by the explosive shock wave (Toksöz *et al.* 1965; Archambeau 1964, 1968; Archambeau & Sammis 1970). In either case the mechanism is stress relaxation; i.e. the reduction of strain energy in the vicinity of the zone of failure. The local reduction of strain energy is accomplished by radiation away from the source region. From a theoretical standpoint, the differences in the two mechanisms is largely geometrical, that is a difference in the shape of the region of failure. However, tectonic radiation must always occur for an explosion in a pre-stressed medium because of the fracture zone created by the shock wave, while the explosion may or may not induce an earthquake (rapid faulting) along a weak zone of high stress concentration, depending on the proximity of such a zone (or zones) to the explosive hypocentre. It appears that induced earthquakes do occur because of stress overloading (see for example, Brune & Pomeroy 1963; Archambeau & Sammis 1970) but that they probably do not occur with every explosive event nor even with most, and when there is such activity it is often in the form of aftershocks quite far removed in time from the main event. Thus, the response to overloading of the medium may often be through plastic deformation preceding (and required for) eventual failure at a later time. Thus in many cases stress radiation due to the shock-induced spherical fracturing applies and, in any case, this process must always account for a part of the anomalous radiation. In this study the nature of the radiation field to be expected from this latter kind of rupture phenomena will be considered and the similar theory for earthquakes, whether induced or 'spontaneous', will be taken up in a separate work.

An approach to problems involving relaxation in prestressed media has been discussed by Archambeau (1964, 1968). In the present study, the particular case involving a spherical fracture zone produced by a shock wave will be discussed in detail in the context of a more concise Green's function formulation of the general problem. The theory is also briefly summarized in a companion study by Archambeau & Sammis (1970) where it is used to examine data from the Bilby explosion.

Aside from the intrinsic interest that this theory may have, it is important to examine the relevance of this mechanism of seismic radiation since it can have important practical applications. For example, it can provide a detailed explanation of the radiation field from explosions and so bears on the problem of distinguishing between earthquakes and underground explosions in general circumstances. It also can be used to deduce source-connected parameters of considerable geophysical interest, in particular the non-hydrostatic stress in the source region, the shock-induced rupture velocity (or velocities) and the dimensions of the fracture zone. Further, since the process involves a net reduction of stress, which is to a large extent, predictable and can be controlled as to time and place, it could serve as a basis of regulating stress accumulation. Hence large destructive earthquakes, which are almost invariably shallow, might be avoided by the release of strain energy in a series of smaller, induced, events.

The following section provides a background discussion of the basic observations and characteristics of the seismic radiation field from underground explosions. Section 3 gives relations for tectonic energy release as a function of prestress and fracture zone parameters. Section 4 takes up the basic formulation for the radiation field, the problem being treated as a generalized initial value problem and formulated in terms of Green's functions. Section 5 gives the evaluation of the general integral relations for specific cases of spherical rupture zone creation.

2. Observational evidence for tectonic energy release

The observations of the field from underground explosions show a radiation of compressional waves which conforms quite closely to that expected from the conversion of a shock wave to a purely compressional elastic wave. This may be described theoretically in terms of an equivalent point compressional source. In this case the observations of the radiation patterns of compressional body phases, when corrected for known structural effects and interpreted in terms of a multi-branched travel-time curve, show a circularly symmetric pattern around the source and the initial motion is compressional. Fig. 1 shows an example of the radiation pattern for compressional phases from the nuclear explosion Bilby which illustrates this circular pattern. The data shown were obtained from a study by Archambeau, Flinn & Lambert (1969).

If the source were purely compressional then we would expect to observe a circularly symmetric radiation pattern for Rayleigh waves and for *SV* body phases like that observed for the compressional body phases. In addition, we would not expect to observe any *SH* polarized waves or Love waves except that minimal amount due to non-linear mode conversion along the path through the medium. However, it is generally the case that the observed Rayleigh wave pattern departs significantly from circular symmetry and that Love waves are strongly excited. The travel times and the amplitude variation with distance for this anomalous part of the radiation field indicates that the origin of these effects is in the source region and so these observations cannot be explained by a non-linear conversion of wave type along the whole path (Toksöz *et al.* 1965; Press & Archambeau 1962). In addition, Press & Archambeau show examples of the elastic field from an explosion in the atmosphere which can be used to gauge the extent of mode conversion in the Earth and while conversion is present, it is clearly inadequate to explain the observations from underground explosions.

Fig. 2 shows an example of the Rayleigh wave radiation pattern for the underground Bilby. The lack of circular symmetry is reasonably apparent from the observations at the stations close to the source. Fig. 3 shows the very large Love waves observed at the same period for this same event. The amplitude of the wholly anomalous Love wave radiation is about half that of the Rayleigh wave field at this period and conforms to a quadrapole radiation pattern. Both of these figures are from a study by Lambert, Flinn & Archambeau (1972) in which it was shown that both the Love wave and Rayleigh wave patterns could be explained by adding a particular double-couple or quadrupole component to the compressional source representing the explosion itself. The insets in Figs 2 and 3 show the theoretical pattern shape for a fixed ratio of excitation of the double couple to the compressional source component (quantitatively measured by F) and for fixed orientation of the double couple. Comparison of these insets with the observed patterns show that the fit to the observations is good. Such a comparison was made at numerous frequencies for the Bilby and Shoal explosions by Lambert *et al.* and all the data could be fitted using this representation of the source. Toksöz *et al.* (1965) conducted a study of the Hardhat, Haymaker and Shoal explosions and obtained similar results.

With the exception of the Hardhat event, the energy in the anomalous radiation field was found to be of the order predicted theoretically by Press & Archambeau (1962) for stress relaxation in the vicinity of a spherical fracture zone, when appropriate values of prestress and fracture zone radius for each event were used. In the case of Hardhat, the anomalous energy was much higher than could be expected from stress relaxation around the explosive fracture zone and it is probable that induced failure along a fault zone also occurred.

A quadrupole source term was shown by Archambeau (1964, 1968) to be an appropriate representation for stress relaxation around a spherical fracture zone created by a shock wave. This therefore agrees with the observation that a double

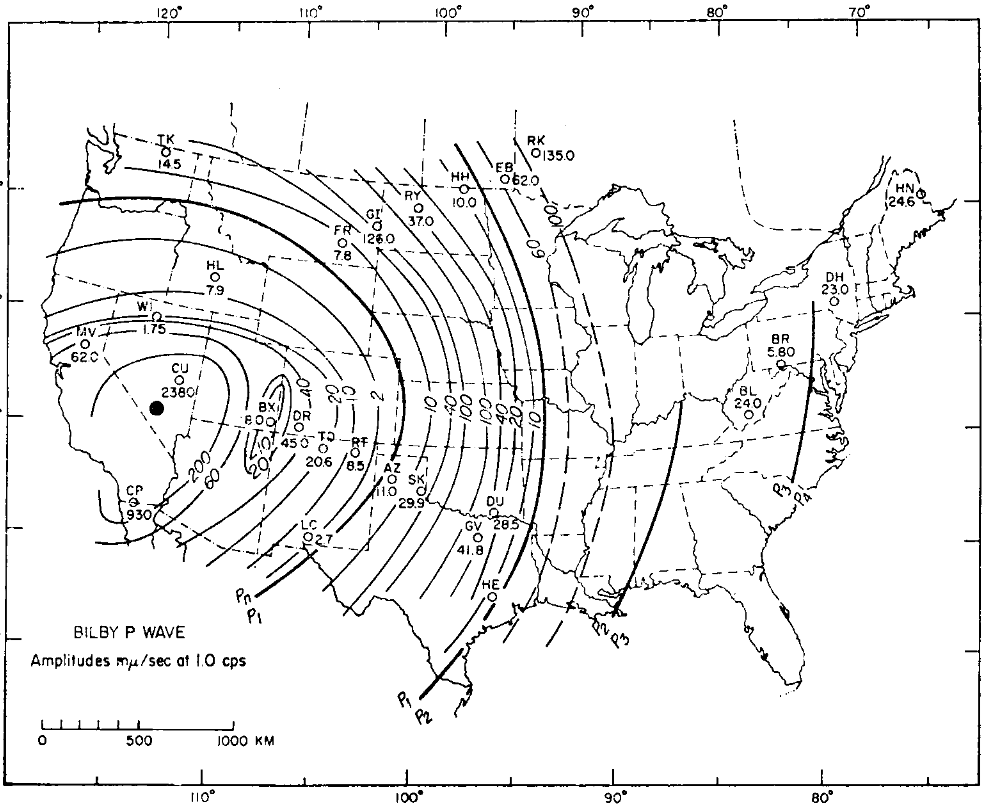

FIG. 1. The radiation pattern of compressional body phases from the underground explosion Bilby, located at the Nevada Test Site. Contours of equal amplitude are shown, the contour interval is not uniform and the amplitudes are appropriate to the spectral amplitude at 1 cps of the first arriving compressional phase. The heavy contours are used to separate different first arrival phases, differentiated on the basis of different paths through the mantle. Each of these phases has a different amplitude dependence with distance so that the heavy contours represent discontinuities of amplitude of the first arrival phase with distance. In each sector corresponding to a given compressional phase type (e.g. P_n, P_1, P_2, etc.) the radiation pattern from this source shows circular symmetry (from Archambeau *et al.* 1969).

couple equivalent provides a fit to the observed anomalous radiation patterns. It was also shown that a quadrupole term was the only multipole term in the solution, while for a propagating unilateral rupture, such as is appropriate for an earthquake, all the higher order multipoles were shown to be excited. As a consequence, a propagating unilateral rupture has a frequency-dependent radiation pattern, which is highly asymmetric at high frequencies. At long periods, such that the wavelength is at least several times larger than the rupture length, the dominant term in the propagating rupture solution is the quadrupole term and so for long wavelength radiation there is no essential difference between the radiation patterns for a fault and that due to the spherical fracture zone. However, the spectrum for an induced earthquake, with the same total energy release, will be shifted to relatively longer periods and so one would expect the spectral peak, or peaks, to occur at longer periods. Thus, if an induced earthquake is the origin of the anomalous radiation from an explosion, we would expect to see more of the energy at the longer periods than if radial fracturing was wholly responsible.

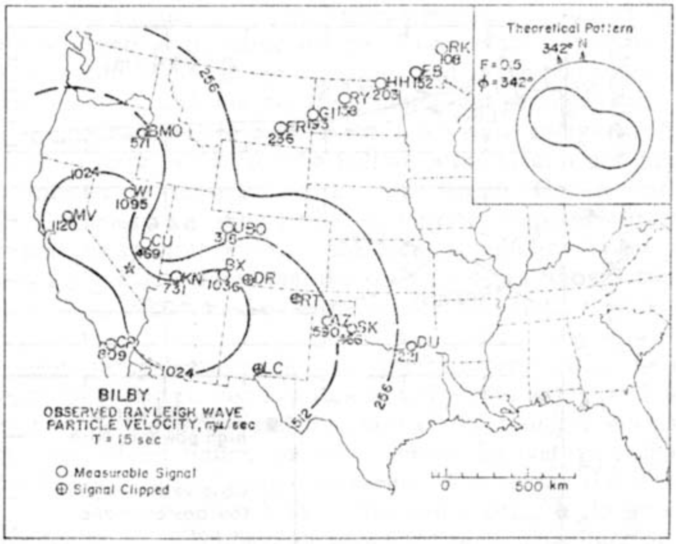

FIG. 2. Rayleigh wave spectral amplitudes observed at a period $T = 15$ s from Bilby explosion. This pattern can be matched by the superposition of monopole and quadrupole sources, corresponding to the pressure pulse from the explosion itself plus a tectonic contribution of quadrupole form due to stress relaxation around the shock induced fracture zone. The inset shows such a theoretical radiation pattern, with $F = 0{\cdot}5$ related to the ratio of energy radiated by the monopole divided by energy from the quadrupole contribution. The angle $\phi = 342°$ is related to the orientation of the quadrupole (from Lambert *et al.* 1972).

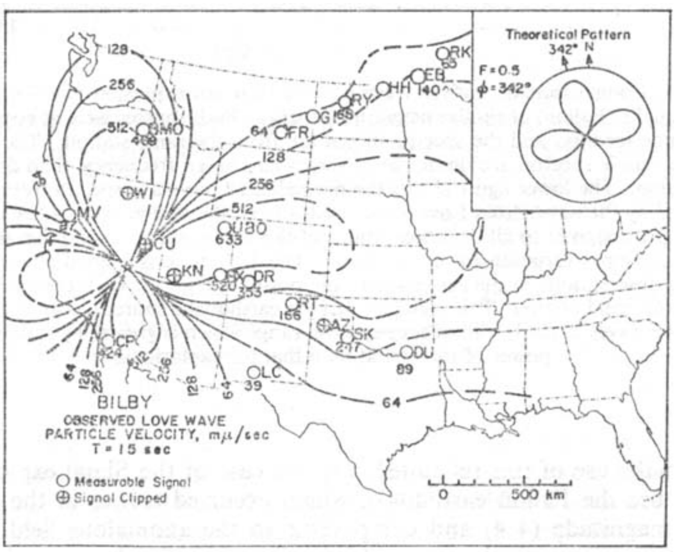

FIG. 3. The Love wave radiation pattern from the Bilby explosion corresponding to the spectral amplitude at a period $T = 15$ s. This pattern can be matched by the same combination of monopole and quadrupole contributions as was used for the fit to the Rayleigh wave pattern shown in Fig. 2. This theoretical pattern is shown in the inset. The quadrupole contribution is totally anomalous in the sense that it is not predicted for an explosive source. Figure from Lambert *et al.* 1972.

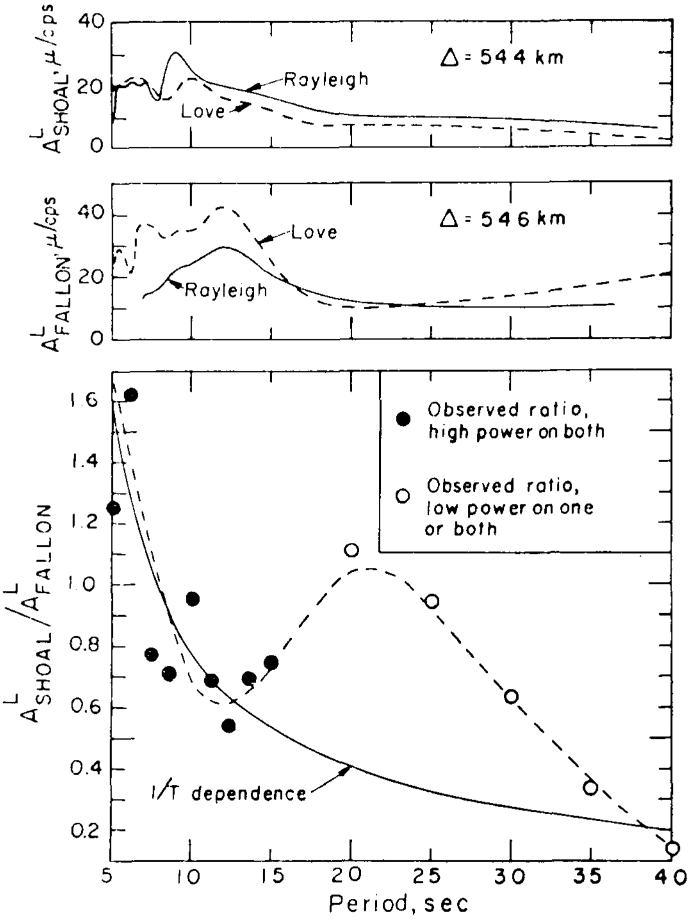

FIG. 4. Comparison of surface wave spectra from an explosion (shoal) and an earthquake (Fallon) of similar magnitude/energy. Both sources were at essentially the same location and the spectra shown are from the same station. The earthquake has a spectral maximum at a somewhat lower frequency than does the explosion. The lower figure shows the normalized Love wave spectra from Shoal divided by the normalized Love wave spectra from the Fallon event. The dashed curve is a rough fit to all the observations of this ratio while the solid line shows a $1/T$ functional variation for comparison. The $1/T$ curve is a good empirical fit to the observations in the range where the power is high for each component of the ratio (solid points). This indicates that the earthquake source has an excitation of Love waves in the low frequency spectral range which is greater by roughly one power of the period than that for explosions.

We can make use of this relationship in the case of the Shoal explosion. In this case we can use the Fallon earthquake which occurred earlier in the same vicinity and was of magnitude (4.4) and comparable to the anomalous field of the Shoal event ($m = 4{\cdot}9$), to test whether an earthquake accounts for the anomalous field from Shoal. If the anomalous field is due to an induced earthquake then the spectral peak of the Love waves, which are totally anomalous, should occur at a period comparable to the corresponding Love wave maxima observed from the Fallon event when both are observed at the same station. If the anomalous field is due to the spherical fracture zone created by the explosion, then we expect to see the spectral peaks shifted to higher frequencies. Fig. 4 shows both the Love and Rayleigh wave spectra for the two events at the same station. The surface wave paths for the two two events are essentially identical and the stress field at the two source sites should have been nearly the same, with that for the Fallon event perhaps higher if anything. Thus, the fault length required to explain the Love wave energy for Shoal would have to be at least as large as that for the Fallon earthquake, if not larger since the energy depends directly on the stress and the fault length. Consequently, the Love waves from Shoal should have long period excitation comparable or greater than that for Fallon and the spectral peaks should occur at equal or longer periods if an earthquake is involved. Fig. 4 shows that this does not occur; the spectral peak in the Love waves from Shoal occurs in the range 7–10 s at this station, while that for the Fallon event is near 13 s.

To see the relationship between the two spectra more clearly, the ratio of the normalized spectrums is also plotted as a function of period in Fig. 4. Each spectrum has been divided by its maximum and the ratio of the resulting normalized Shoal spectrum to the normalized Fallon spectrum shows the relative excitation of the fundamental mode Love waves for the two events. The curves, therefore, show the ratio of energy, velocity or displacement for the two events. If the anomalous field from Shoal was due to an earthquake of about the same magnitude as Fallon, the ratio would be near unity at all periods. A curve showing a $1/T$ dependence is plotted for comparison and it appears that the observations show this kind of dependence where there is high power in both spectrums. For periods from around 15 to 25 s both spectrums have minimums and low power. This may account for the deviation in the observed curve from a $1/T$ variation in this range, but in any case the variation at longer periods (out to 40 s) is toward lower relative power in the Shoal spectrum, and tending to approach the $1/T$ dependence. Roughly speaking, this kind of variation is what would be expected from a comparison of the radiation from a spherical fracture zone with that from a linear zone of failure appropriate to an earthquake.

In order to illustrate an even more extreme case, Fig. 5 shows the Love and Rayleigh wave spectrums from the Bilby event (magnitude 5.8) with the comparable spectrums from Fallon. The paths for these waves are essentially the same, although that for Fallon is slightly longer, by about 170 km. The anomalous field associated with the Bilby event would require an induced earthquake larger than Fallon and since the events were in the same area with roughly the same stress levels, this would require a significantly longer fault. Therefore, the Love waves for Bilby should peak at a longer period and the amplitudes at long periods should be considerably greater. The Love spectrum in Fig. 5 shows that this is not the case, the Love waves for Bilby again peaking near 10 s while the spectrum for Fallon has a maximum in the range from 15 to 30 s. The plot of the ratio of the normalized amplitudes shows this most clearly. In this comparison the power estimates are good at the long periods and the fit to a $1/T$ dependence appears good. The scatter at short periods is probably related to the difficulty of obtaining good amplitude estimates at low power and since we are dealing with a ratio, the variance is very high. Even so, the observations at periods below 15-s period do suggest relatively much more energy in the Bilby spectra at short periods and, coupled with the better estimates at longer periods, show that the source dimensions appropriate to the anomalous radiation from Bilby were considerably smaller than those appropriate to a fault which would give the required total energy release. The observations strongly imply an explanation in terms of stress release from the smaller region around the spherical fracture zone.

More precisely, Archambeau & Sammis (1970), using the theory to be developed in this study, have shown that the amplitude as well as the pattern shape of the anamalous Love wave radiation observed from Bilby are consistent with stress relaxation around a shock induced fracture zone when reasonable values of prestress (70 b) and fracture zone radius (420 m) are used.

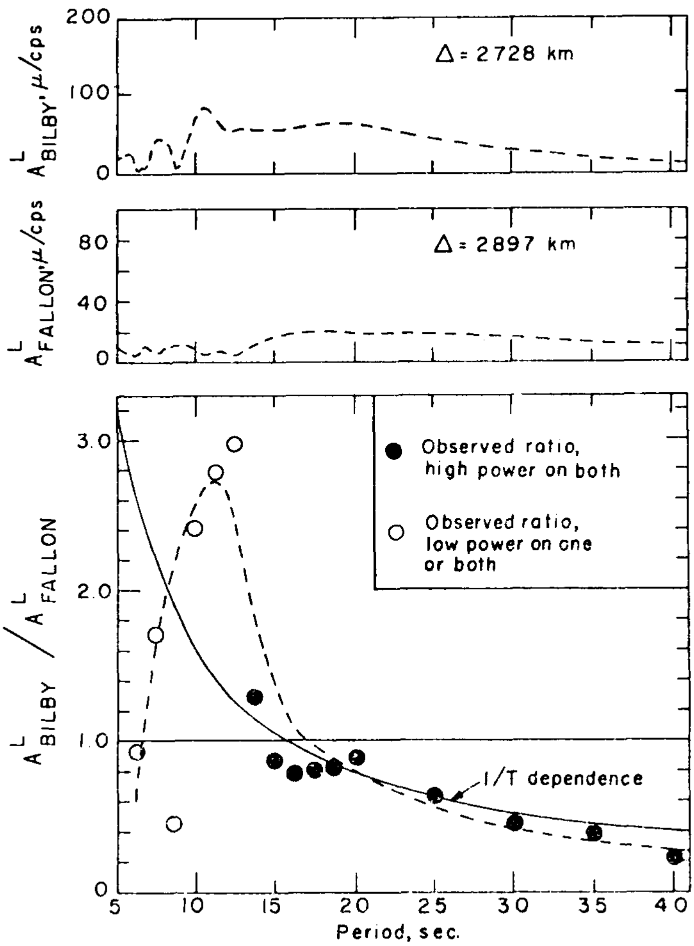

FIG. 5. Love wave spectra for the explosion Bilby and the Fallon earthquake observed at the same station so that the paths of propagation and the distances to the sources are nearly the same. The Bilby explosion was of larger seismic magnitude than the earthquake, so that the excitation of long period waves is good in both cases. In spite of the larger magnitude of the explosion, the earthquake shows a spectral peak at lower frequency. The lower figure shows the ratio of the normalized spectra and in this case the power at long periods for both sources is adequate, so that the long-period ratio is meaningful. Again, as in Fig. 4, a $1/T$ dependence appears to fit the observations implying a different excitation by approximately one power of the period in this period range.

Thus, in at least three (out of four) carefully analysed cases it appears that simple stress relaxation in the spherical fracture zone created by the explosion can reasonably explain the observations of body and surface wave radiation patterns, and the total energy radiated. For Bilby at least, the amplitudes of the anomalous surface waves could also be explained. In all cases it appears that when underground explosions are detonated in tectonic areas, one can expect stress release due to creation of a high symmetry fracture zone and in some cases additional tectonic energy release due to induced faulting.

A different test of whether tectonic energy release is responsible for the anomalous field is available from the observations of the explosion Salmon, which was detonated in a salt dome. In this case one would expect little or no prestress in the salt itself and so little or no anomalous radiation. A study by Archambeau, Flinn & Lambert (1966) shows this to be the case. In particular, both the compression body wave radiation patterns and the Rayleigh wave pattern were circular and there was little or no Love wave radiation observed.

3. Energy from stress relaxation

Let a zero superscript denote the equilibrium stress, strain and displacement in a prestressed medium with fixed boundaries. Similarly, let the superscript (1) denote these quantities after the introduction of a new boundary within the medium. In the case at hand the new boundary is the surface enclosing the fracture zone created by an explosion. This region will be taken to be spherical. Finally, define differences of stress, strain and displacement, so that

$$\sigma_{ij}^* = \sigma_{ij}^{(0)} - \sigma_{ij}^{(1)}$$

for example. Now with **f** denoting all the static forces on the medium under consideration, we have the equilibrium equations

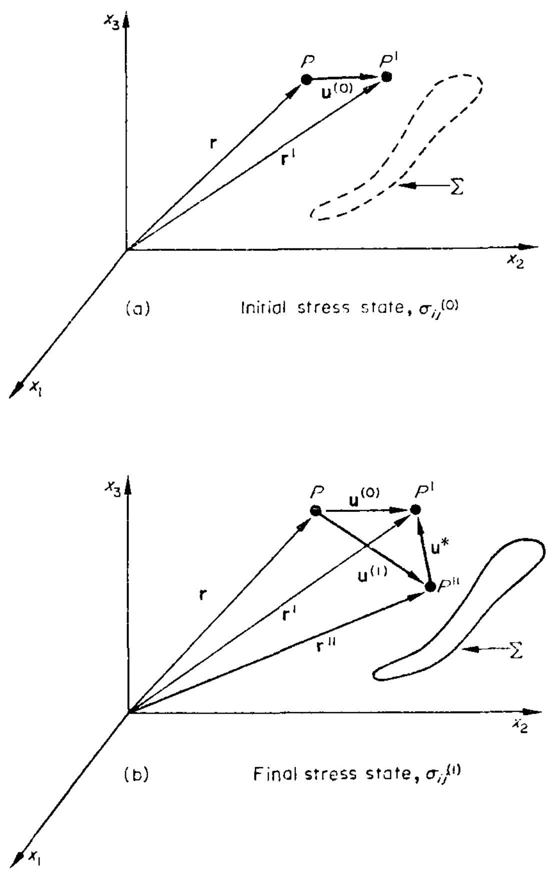

FIG. 6. (a) Schematic representation of the initial stress state of the medium where $\mathbf{u}^{(0)}$ denotes the displacement of material points from the unstressed state and Σ denotes the surface of failure which will be formed. (b) Representation of the final stress state outside the surface Σ, after the formation of the rupture surface Σ, with $\mathbf{u}^{(1)}$ the displacement of the final state from the zero stress (reference) state. $\mathbf{u}^*$ represents the change in the field from initial to final states, measured from the final state.

$$\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} + f_i = 0 \tag{3.1}$$

and boundary conditions

$$[\sigma_{ij}^{(0)} n_j] = 0; \quad r \in B \tag{3.2}$$

$$[u_i^{(0)}] = 0$$

where $\sigma_{ij}^{(0)}$ is the initial stress state of the medium and $u_i^{(0)}$ the displacement. Fig. 6 shows the relationship of the initial state variables to those in the final state, in schematic form.

Here the bracket notation will be used to denote the change in traction and displacement across the fixed boundaries, B of the medium. Hence,

$$[u_i] = u_i(\mathbf{r}+\boldsymbol{\delta}) - u_i(\mathbf{r}-\boldsymbol{\delta}),$$

with $\mathbf{r}+\boldsymbol{\delta}$ and $\mathbf{r}-\boldsymbol{\delta}$ displaced infinitesimally on opposite sides of the boundary surface B. In addition, the summation convention will be used throughout.

Similarly, the final stress state of the medium is $\sigma_{ij}^{(1)}$, and is to first order:

$$\frac{\partial \sigma_{ij}^{(1)}}{\partial x_j} + f_i = 0 \tag{3.3}$$

with the boundary conditions of (3.2) applied to $\sigma_{ij}^{(1)}$ and $u_i^{(1)}$ on the fixed boundaries B and on the new boundary Σ. The additional boundary condition associated with Σ requires that $\sigma_{ij}^{(1)}$ be different than $\sigma_{ij}^{(0)}$. In view of the identical form of (3.1) and (3.3) however, we can subtract the equations and thereby eliminate the static force term acting on the medium and obtain an equation involving the stress difference σ_{ij}^*. We have, to first order

$$\frac{\partial \sigma_{ij}^*}{\partial x_j} = 0. \tag{3.4}$$

Thus the stress change satisfies a homogeneous equation, without source terms. In addition, since $\sigma_{ij}^{(1)}$ must approach $\sigma_{ij}^{(0)}$ at large distances from the (closed) boundary surface Σ, then σ_{ij}^* vanishes at large distances from Σ.

Introducing a displacement u_i^* and strain e_{ij}^*, where as usual

$$\sigma_{ij}^* = \lambda e_{kk}^* \delta_{ij} + 2\mu e_{ij}^*$$

and

$$e_{ij}^* = \frac{1}{2}\left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i}\right)$$

then we have from (3.4)

$$(1-2\sigma)\,\nabla^2 \mathbf{u}^* + \nabla(\nabla\cdot\mathbf{u}^*) = 0 \tag{3.5}$$

where $\sigma = \lambda/[2(\lambda+\mu)]$ is Poisson's ratio.

In later developments it is advantageous to make use of potentials χ_α, $\alpha = 1, 2, 3, 4$ which will be defined as the Cartesian components of a four vector, where

$$\chi_k^* = \tfrac{1}{2}\delta_{knm}\frac{\partial u_m^*}{\partial x_n}; \qquad \chi_4^* = \frac{\partial u_l^*}{\partial x_l}. \tag{3.6}$$

Throughout this paper Roman subscripts and superscripts will run over the range 1, 2, 3 while Greek subscripts will always be over the range 1 to 4. The potentials of (3.6) are seen to be the components of the rotation vector and the dilatation, respectively. Taking the curl of (3.5) shows that the χ_k^* are harmonic and taking the divergence of (3.5) verifies that χ_4^* is also harmonic; hence

$$\nabla^2 \chi_\alpha^* = 0. \tag{3.7}$$

The strain energy density is $\frac{1}{2}(e_{ij}^{(0)} \sigma_{ij}^{(0)})$ initially and in the final equilibrium state, in the presence of Σ, it is $\frac{1}{2}(e_{ij}^{(1)} \sigma_{ij}^{(1)})$. Thus the change in the total strain energy throughout the medium is

$$\delta W = \frac{1}{2} \int_V \{\sigma_{ij}^{(0)} e_{ij}^{(0)} - \sigma_{ij}^{(1)} e_{ij}^{(1)}\} \, d\mathbf{r} \tag{3.8}$$

where V is the total volume of the medium. If Σ is a closed surface then let V_0 be the volume contained within Σ and V_1 the volume outside Σ, so $V = V_0 + V_1$. If Σ is not closed then $V_0 = 0$. Within V_0, $\sigma_{ij}^{(1)}$ and $e_{ij}^{(1)}$ will be taken to correspond to the hydrostatic stress and strain under the assumption that Σ encloses a region of vanishing shear modulus. For fine scale fracturing this should be a good assumption. This need not necessarily be assumed in order to calculate the value of (3.8), but the assumption seems appropriate for the case of shock induced rupture.

Thus since the medium goes from one potential energy state to another due to the formation of the rupture volume V_0, it follows that energy will be released or absorbed depending on the sign of (3.8). Press & Archambeau (1962) have shown that (3.8) can be put in the form

$$\delta W = \frac{1}{2} \int_{V_0} \sigma_{ij}^{(0)} e_{ij}^{(0)} \, d\mathbf{r} + \frac{1}{2} \int_{V_1} \sigma_{ij}^* e_{ij}^* \, d\mathbf{r} + \int_V f_i u_i^* \, d\mathbf{r} \tag{3.9}$$

when the rupture zone is considered a cavity (not necessarily spherical) where the surface tractions vanish. The first term is the potential strain energy originally stored within the cavity zone, the second corresponds to the change in the potential energy in the medium surrounding the cavity and the last is the work done by or against the body and tectonic forces. The precise definition of what is meant by a 'tectonic' force is given in Appendix I along with other general considerations. The second term is generally used to estimate the energy released as seismic radiation, for example, by Press & Archambeau (1962), Knopoff (1958), Toksöz *et al.* (1965) and by Archambeau & Sammis (1970). This relation is not precisely what is needed however since we wish to determine the energy release due to the formation of a rupture zone where the material assumes the properties of a fluid. Thus instead of a void cavity we will consider a fluid filled volume, V_0, enclosed by a surface, Σ. In this case we have, from Appendix I;

$$\delta W = \delta W_0 + \frac{1}{2} \int_{V_1} \sigma_{ij}^* e_{ij}^* \, d\mathbf{r} + \int_{V_1} f_i u_i^* \, d\mathbf{r} + \int_\Sigma \sigma_{ij}^{(1)} \Delta u_i(\Sigma^-) n_j \, dS \tag{3.10}$$

where

$$\delta W_0 = \int_{V_0} [\sigma_{ij}^{(0)} e_{ij}^{(0)} - \sigma_{ij}^{(1)} e_{ij}^{(1)}] \, d\mathbf{r}.$$

This result is, in fact, valid whether the material in V_0 is fluid or not, so long as the elastic properties in V_0 are different than those in V_1.

The first term in (3.10) is analogous to the first term in (3.9), except in this case $\sigma_{ij}^{(1)}$ and $e_{ij}^{(1)}$ do not vanish in V_0 and so the contribution to the change in energy from V_0 is not the total energy originally present, but only the change in the strain energy. The second and third terms are essentially the same as those in (3.9) except that contributions from V_0 are not included. The fourth term represents an additional contribution to δW, where

$$\Delta u_i \equiv u_i^* = (u_i^{(0)} - u_i^{(1)}); \quad \mathbf{r} \in \Sigma^-$$

is the relaxation displacement on the rupture surface Σ as measured in the elastic medium, so that Σ^- is the surface approached from within V_1. We note that $u_i^{(1)}$ and therefore u_i^* need not be continuous across the rupture boundary Σ, and clearly need not be when the medium within V_0 is fluid.

The nature of the contribution from this term can be seen by noting that for an ideal fluid.

$$\sigma_{ij}^{(1)} = -p_0 \delta_{ij}, \quad \mathbf{r} \in \Sigma.$$

That is the final stress on Σ is a hydrostatic pressure p_0. Further $\sigma_{ij}^{(1)} n_j$ is continuous across Σ, so $[\sigma_{ij}^{(1)} n_j] = 0$ on Σ. Thus this additional term is

$$+ \oint_\Sigma \sigma_{ij}^{(1)} \Delta u_i n_j \, dS = - \oint_\Sigma p_0 \Delta u_i n_i \, dS \equiv - \oint_\Sigma p_0 \Delta \mathbf{u} \cdot \hat{\mathbf{n}} \, dS$$

and this is just the work done against the hydrostatic pressure, for example in any expansion of the material.

Thus for a fluid rupture zone of arbitrary shape

$$\delta W = \delta W_0 + \frac{1}{2} \int_{V_1} \sigma_{ij}^* e_{ij}^* \, d\mathbf{r} + \int_{V_1} f_i u_i^* \, d\mathbf{r} - \oint_\Sigma p_0 \Delta u_i(\Sigma^-) n_i \, dS \tag{3.11}$$

with p_0 the fluid pressure in the rupture zone. The first two terms in (3.11) are positive indicating a release of potential energy. The other two terms may be positive or negative depending on the signs of the inner products $f_i u_i^*$ in V_1 and $\Delta u_i n_i$ on Σ^-. In this discussion no consideration has been given to the irreversible processes of material failure and it appears, as noted in Appendix I, that the first term in (3.11) would be part of the energy required to balance the energy requirements for fracture and/or phase change, such as melting. Archambeau (1968) discusses this point in the general case. In the case of an explosion, the chemical or nuclear energy release would contribute to these irreversible processes as well. In addition, of course, the explosive reaction produces the shock wave which eventually is converted to an elastic wave accounting for a part of the seismic field. The energy term which accounts for most of the (anomalous) seismic radiation due to changes in the equilibrium stress state of the medium is the second, the relaxation energy. The third term is small compared to the second. In the non-linear region V_0 such a term could be large and contributes to the failure process; however, in this formulation it has been included in δW_0. The size of the final term relative to the relaxation energy depends on some knowledge of magnitude of $\Delta \mathbf{u} \cdot \hat{\mathbf{n}}$, which is the relaxation displacement normal to the failure surface.

In particular, if the relaxation integral is transformed to a surface integral so that

$$\frac{1}{2} \int_{V_1} \sigma_{ij}^* e_{ij}^* \, d\mathbf{r} = \frac{1}{2} \int_{V_1} \frac{\partial}{\partial x_j} (\sigma_{ij}^* u_i^*) \, d\mathbf{r} = \frac{1}{2} \int_{\Sigma^-} \sigma_{ij}^* u_i^* n_j \, dS$$

then the terms can be compared directly. Since the tractions $\sigma_{ij}^* n_j$ include forces both normal to and along the rupture surface (shearing), then all components of u_i^* contribute to the inner product arising from this integral. On the other hand only the normal components contribute to the final integral in (3.11) and if we take the initial shear stress in $\sigma_{ij}^{(0)}$ to be comparable to the hydrostatic pressure at shallow depths, then we expect the shearing components of the relaxation displacements

u_i^* to be much larger than the (compressional) components. Therefore the inner product in the integrand for the relaxation term would, under these conditions which are appropriate for explosions, be correspondingly larger than the integrand in the final surface integral of (3.11).

However, if, as can be the case in an explosion, the final stress field $\sigma_{ij}^{(1)}$ contains the effects of overpressure due to high fluid temperature in V_0 relative to the solid material in V_1 then the total pressure is $p = p_0 + p_1$ where p_0 is the hydrostatic pressure and p_1 is the overpressure. Since thermal conduction losses are so slow compared to dynamical effects in elasticity, including the time it takes to reach a state of near elastic equilibrium, we can treat p_1 as being static in so far as elastic effects are concerned. It is therefore useful to retain the general form (3.10) wherein the surface tractions $\sigma_{ij}^{(1)} n_j$ on Σ are left unspecified.

In view of the meaning and relative size of the terms in (3.10), an estimate of the tectonic energy released, E_R, is given by the second and last terms appearing in (3.10). That is,

$$E_R \simeq \frac{1}{2} \int_{V_1} \sigma_{ij}^* e_{ij}^* \, d\mathbf{r} + \int_{\Sigma} \sigma_{ij}^{(1)} u_i^*(\Sigma^-) n_j \, dS.$$

Reducing the volume integral over V_1 to a surface integral over Σ by the methods employed in the Appendix I we have

$$E_R \simeq \int_{\Sigma} \left(\frac{\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}}{2} \right) u_i^*(\Sigma^-) n_j \, dS. \tag{3.12}$$

This shows that we may interpret the stress relaxation in terms of an equivalent dislocation. In particular the appropriate stress acting on the dislocation is the mean stress $(\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)})/2$ and the displacement discontinuity should be

$$\Delta u_i = u_i^*(\Sigma^+) = u_i^{(0)} - u_i^{(1)}.$$

Clearly this is the work done in creating such a dislocation and we have verified that it is just equal to the energy released by stress relaxation in the volume V_1. Steketee (1958) has discussed the use of dislocations in this context, this result shows explicitly what values of stress and displacements discontinuity are proper. Archambeau & Minster (1972) consider this question in more general terms, with applications to earthquakes in particular. In any case, it is clear that the treatment just given is quite general and that it applies in an entirely parallel fashion to any kind of relaxation source.

The computation of the energy release E_R can be easily accomplished using (3.12) once the boundary value problem for a spherical inclusion in a prestressed medium has been obtained. Ususally solutions are given in terms of the stresses $\sigma_{ij}^{(1)}$, for example by Landau & Lifshitz (1959). In this case a convenient estimate of E_R is obtained from (3.10) in the form of a volume integral as

$$E_R \simeq \int_{V_1} \left(\frac{\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}}{2} \right) e_{ij}^* \, d\mathbf{r} \tag{3.13}$$

where this estimate for E_R now contains all three of the last terms in (3.10). Knowledge of $\sigma_{ij}^{(1)}$ for a given prestress can be used to obtain E_R from (3.13) since e_{ij}^* can be calculated from its definition, using the stress–strain relations.

As it happens Eshelby (1957) calculates an energy equivalent to the right-hand side of (3.13), which he calls E_{int}, the interaction energy. Thus from Eshelby's results we have

$$E_R \simeq \tfrac{1}{2}(\tfrac{4}{3}\pi R^3)\left\{\frac{A}{9k}\,p^{(0)}p^{(0)}+\frac{B}{2\mu}\,'\sigma_{ij}^{(0)}\,'\sigma_{ij}^{(0)}\right\} \tag{3.14}$$

in the case of a spherical inclusion of radius R, with elastic moduli μ_1, λ_1, in the presence of a general uniform prestress $\sigma_{ij}^{(0)}$. Here

$$A = \frac{k-k_1}{k}\left(\frac{4\mu+3k}{4\mu+3k_1}\right); \quad k = \lambda+\tfrac{2}{3}\mu, \quad k_1 = \lambda_1+\tfrac{2}{3}\mu_1$$

$$B = \frac{\mu_1-\mu}{(\mu-\mu_1)\,\beta-\mu}, \quad \beta = \frac{2}{15}\left(\frac{4-5\sigma}{1-\sigma}\right)$$

with σ denoting Poissons ratio. The stresses are written in terms of their scalar and deviatoric parts, so

$$\sigma_{ij}^{(0)} = \tfrac{1}{3}p^{(0)}\delta_{ij}+{}'\sigma_{ij}^{(0)}.$$

Thus $\qquad \sigma_{ii}^{(0)} = p^{(0)}$ and ${}'\sigma_{ij}^{(0)} = \sigma_{ij}^{(0)}-\tfrac{1}{3}p^{(0)}\delta_{ij}.$

We are interested in the case of a fluid inclusion, so $\mu_1 = 0$. In this case the energy is

$$E_R \simeq \left(\frac{1}{2}\right)\frac{15(1-\sigma)}{7-5\sigma}\left(\frac{4}{3}\pi R^3\right)\left(\frac{'\sigma_{ij}^{(0)}\,'\sigma_{ij}^{(0)}}{2\mu}\right)\left[1+\frac{A}{B}\left(\frac{2\mu}{9k}\right)\frac{p^{(0)}p^{(0)}}{'\sigma_{ij}^{(0)}\,'\sigma_{ij}^{(0)}}\right]. \tag{3.15}$$

The second term in the brackets is the ratio of energy derived from the hydrostatic field compared to that from the non-hydrostatic prestress field. It is of considerable interest since it gives an estimate of the relative energy in compressional waves compared to shear waves from relaxation. Using $\mu \sim \lambda \sim \lambda'$ as representative and denoting this ratio as $E_R^{(p)}/E_R^{(s)}$ we find

$$E_R^{(p)}/E_R^{(s)} \simeq \tfrac{1}{30}[p^{(0)}p^{(0)}/'\sigma_{ij}^{(0)}\,'\sigma_{ij}^{(0)}]. \tag{3.16}$$

Thus at modest depths (near 1 km) p_0 is around 300 b and since estimated values of $'\sigma_{ij}^{(0)}$ at shallow depths have been of the order of 100 b, for example by Chinnery (1964), Kaula (1963), Wyss (1969), then it is to be expected that

$$E_R^{(p)}/E_R^{(s)} \simeq \tfrac{1}{10}$$

for explosions. Hence the compressional energy release by the relaxation process under consideration is an order of magnitude less than the shear energy release. This, as was previously pointed out, is observed to be the case for explosions.

Since $E_R^{(p)}$ is so small compared to $E_R^{(s)}$ we could neglect the second term in (3.15). This gives for a total energy estimate

$$E_R \simeq \left(\frac{1}{2}\right)\frac{15(1-\sigma)}{7-5\sigma}(\tfrac{4}{3}\pi R^3)\left(\frac{'\sigma_{ij}^{(0)}\,'\sigma_{ij}^{(0)}}{2\mu}\right) \tag{3.17}$$

a result essentially the same as is obtained by considering a cavity in a pure shear prestress field. Thus it has been shown that in spite of the complexities introduced by the scalar pressure field, it is valid to estimate the energy using (3.17). Further we find that $E_R^{(p)} \simeq \tfrac{1}{10}E_R$.

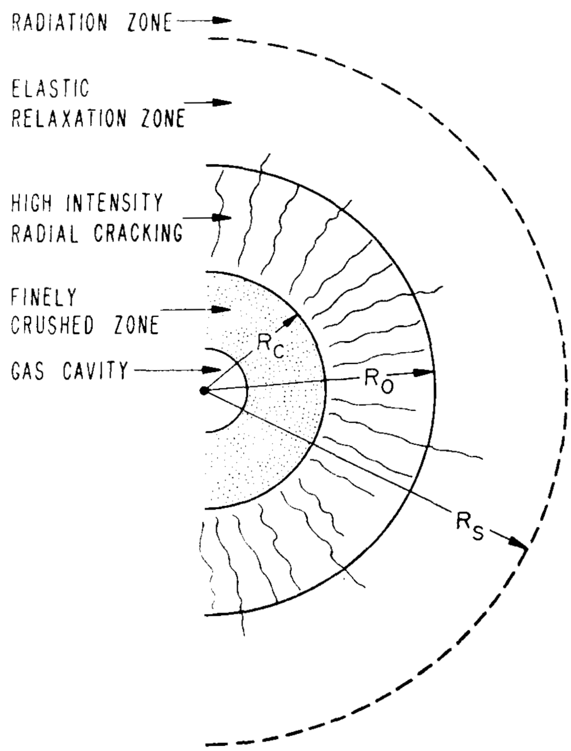

FIG. 7. Schematic representation of the radial zone of material and field behaviour around a large explosion in a prestressed medium. R_c and R_0 are radii enclosing zones of failure and vary with explosive yield, prestress and medium type while R_s is the radius within which most of the relaxation effects occur; for uniform prestress R_s can be taken to be about four times R_0.

Since this is the case, we revert to the simpler solution for a cavity in a pure shear field for all additional calculations. This solution has been given by Landau & Lifshitz (1959) in terms of the change in displacement u_i^*, as

$$u_i^* = \frac{3R^2(1-\sigma)}{\mu(7-5\sigma)}\,\sigma_{ik}^{(0)}\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) - \frac{R^5}{4\mu(7-5\sigma)}\,\sigma_{kl}^{(0)}$$

$$\times\frac{\partial^3}{\partial x_i\,\partial x_k\,\partial x_l}\left(\frac{1}{r}\right) - \frac{5R^3}{4\mu(7-5\sigma)}\,\sigma_{kl}^{(0)}\frac{\partial^3}{\partial x_i\,\partial x_k\,\partial x_l}(r) \qquad (3.18)$$

and σ_{ij}^* and e_{ij}^* are easily calculated from this result. In this case the energy release is obtained from (3.10) where only the second term need be retained. Thus

$$E_R \simeq \frac{1}{2}\int_{V_1} \sigma_{ij}^*\, e_{ij}^*\, d\mathbf{r}. \qquad (3.19)$$

Finally since the energy is independent of the co-ordinate system, we can arrange to make $\sigma_{12}^{(0)} = \mu S_{12}$ with all other $\sigma_{ij}^{(0)}$ equal zero by a rotation of co-ordinates.

For the purpose of this study the formation of a fracture zone around an explosion can be viewed in very simplified terms. Fig. 7 indicates, in a gross way, the state of the medium after an explosion. The inner gas filled cavity is created by compression and vaporization of the surrounding rock, the zone of radius R_c is made up of pulverized material, fractured to the extent that it has little shear strength. This zone is created by the shock wave at a rate characterized by the appropriate shock wave velocity. The radially cracked region can be enclosed for the most part by a spherical region of region of radius R_0, and is probably due to a combination of high static pressure, the existing prestress of the material and the dynamic overloading from the large amplitude pressure wave from the explosion. The rate at which this zone is created is uncertain and could be controlled by the dynamic overloading, in which case it is higher than or equal to the velocity of compressional waves in the medium, or it could be controlled by the static loading in which case the fracture rate would be similar to a spontaneous rupture rate and thus less than the shear velocity of the medium. We will consider both possibilities in subsequent developments. In any case this zone is intensely fractured and would have a low effective shear strength. It is possible, depending on how this zone is formed and particularly on the rate of formation, that some of the initially stored strain energy will show up in the radiation field. On the other hand the energy due to prestress from the region inside the radius R_c will almost certainly be dissipated in the processes of fine scale fracturing. Hence we may wish to include the energy initially stored in the range $R_c < r < R_0$ in an estimate for E_R, but will certainly exclude that initially stored in the range $0 < r < R_c$. In terms of the previous discussion of energy, this means that the region V_0, here associated with the spherical zone of radius R_0, would be broken into two parts with a contribution to the radiation field from within $R_c < r < R_0$ included.

We will assume that a slow rupture velocity, v_R in the region $R_c < r < R_0$ will lead to some contribution to the energy radiated. Thus with $\sigma_{12}^{(0)} = \mu S_{12}$ and $\sigma_{ij}^{(0)} = 0$ for $i, j \neq 1, 2$, we have as an upper bound on the total energy radiated, using (3.19):

$$E_R'' = \int_0^{2\pi} \int_0^{\pi} \left[\int_{R_c}^{R_0} \left(\frac{\mu S_{12}^2}{2} \right) r^2\, dr + \int_{R_0}^{\infty} \sigma_{ij}^* e_{ij}^* r^2\, dr \right] \sin\theta\, d\theta\, d\phi$$

where σ_{ij}^* and e_{ij}^* are found from (3.18). If the velocity v_R is greater than v_S in $R_c < r < R_0$ then we will assume the energy to be lost to the radiation field. Thus as a lower bound we have

$$E_R' = \int_0^{2\pi} \int_0^{\pi} \int_{R_0}^{\infty} \sigma_{ij}^* e_{ij}^* r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Press & Archambeau (1962) give e_{ij}^* for this case and the integrations yield

$$E_R'' = \frac{\mu S_{12}^2}{2} \left(\tfrac{4}{3}\pi R_0^3\right) \left[\left\{ 1 - \left(\frac{R_c}{R_0} \right)^3 \right\} + f\left(\frac{\mu}{\lambda} \right) \right] \tag{3.20}$$

$$E_R' = \frac{\mu S_{12}^2}{2} \left(\tfrac{4}{3}\pi R_0^3\right) f\left(\frac{\mu}{\lambda} \right) \tag{3.21}$$

with

$$f\left(\frac{\mu}{\lambda} \right) = \frac{2[3 + 8(\mu/\lambda)]}{[9 + 14(\mu/\lambda)]}.$$

For the possible range of μ/λ we have

$$\tfrac{2}{3} \leqslant f\left(\frac{\mu}{\lambda} \right) \leqslant \tfrac{8}{7}$$

and for $\mu \simeq \lambda$, as is common

$$f\left(\frac{\mu}{\lambda}\right) = \frac{22}{23}.$$

It seems reasonable to expect that most of the energy release will occur within a volume of radius R_S, a few times larger than R_0. If this is the case we can approximate the source volume as a finite spherical region of radius R_S, and this will simplify the dynamical calculations of the radiation field to follow. In addition it is important to verify that the relaxation effects are local since the heterogeneity of the Earth and the variability of the stress could lead to a violation of the implicit assumptions of uniform prestress in the 'source region'. The contention of local relaxation seems quite plausible in view of the rapid decrease in σ_{ij}^* with increasing r. As obtained from (3.18) it is of order $1/r^3$. To investigate this more quantitatively we can compute E_R', for example, as a function of R_S and compare this to the limiting case $R_S \to \infty$, which has already been computed. Thus we form

$$E_R'(R_S) = \int_0^{2\pi} \int_0^{\pi} \int_{R_0}^{R_S} \sigma_{ij}^* \, e_{ij}^* \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and obtain

$$E_R'(R_s) = \tfrac{1}{2}\mu S_{12}^2(\tfrac{4}{3}\pi R_0^3)\left[f\left(\frac{\mu}{\lambda}\right) - f_1\left(\frac{\mu}{\lambda}\right)\left(\frac{R_0}{R_s}\right)^3 - f_2\left(\frac{\mu}{\lambda}\right)\left(\frac{R_0}{R_s}\right)^5 - f_3\left(\frac{\mu}{\lambda}\right)\left(\frac{R_0}{R_s}\right)^7\right]$$

with

$$f_1\left(\frac{\mu}{\lambda}\right) = \frac{10[27+66(\mu/\lambda)+44(\mu/\lambda)^2]}{(9+14[\mu/\lambda])^2}$$

$$f_2\left(\frac{\mu}{\lambda}\right) = \frac{216[1+(\mu/\lambda)]^2}{(9+14[\mu/\lambda])^2}$$

$$f_3\left(\frac{\mu}{\lambda}\right) = -2f_2\left(\frac{\mu}{\lambda}\right); \quad f\left(\frac{\mu}{\lambda}\right) = f_1 + f_2 + f_3.$$

Now it is a simple matter to compute the error we make in taking the source radius to be a particular value of R_s rather than an extremely large value. We see from Fig. 8 that if R_s is 3 to 4 times R_0, then the error is small. This also allows us to use a uniform prestress for $\sigma_{ij}^{(0)}$ and to associate it in a meaningful way with the average stress in the region within R_s. If R_s were very large it would be hard to envision what relation an assumed uniform stress $\sigma_{ij}^{(0)}$ would have to the real stress state of the material.

However, since R_s is of the same order as R_0 and relatively small compared to geologic features which could significantly modify the mean stress field, we can take the prestress within R_s to be uniform and representative of the average stress field in this region.

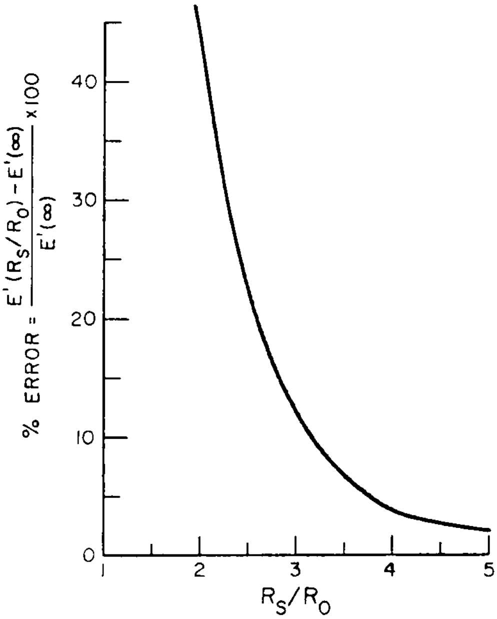

FIG. 8. Variation of energy release from the prestressed medium around an explosive induced fracture zone of radius R_0 as a function of the relaxation zone radius R_s. The energy $E'(R_S/R_0)$ is the calculated strain energy change within the spherical zone from R_0 to R_S and is equal to the energy radiated from this zone. $E'(\infty)$ is the total energy released when R_S/R_0 approaches infinity, assuming uniform prestress. The curve shows the per cent radiated from a relaxation zone of radius R_s. The curve shows that over 95 per cent of the energy comes from within the zone $R_S \leqslant 4R_0$.

4. Dynamical relaxation

Consider a growing spherical rupture in a prestressed medium. For dynamical equilibrium we have

$$\frac{\partial \sigma_{ij}}{\sigma x_j} + f_i = \rho \frac{\partial^2}{\partial t^2} u_i \tag{4.1}$$

where $\sigma_{ij}(\mathbf{r}, t)$ is the total stress field. The stress obeys the continuity condition

$$[\sigma_{ij} n_i] = 0, \qquad r \in B$$

on all boundaries B of the medium and in particular on any growing rupture surface. Introducing relative stress and displacements defined by

$$\tau_{ij}(\mathbf{r}, t) = \sigma_{ij}(\mathbf{r}, t) - \sigma_{ij}^{(1)}(\mathbf{r})$$

$$\mathbf{y}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t) - \mathbf{u}^{(1)}(\mathbf{r})$$

and separating the body force into static and dynamic parts

$$f_i(\mathbf{r}, t) = f_i(\mathbf{r}) + F_i(\mathbf{r}, t)$$

with $F_i(\mathbf{r}, t)$ representing dynamical forces equivalent to the explosion itself in this case, then we have, using the static equations (3.1) and (3.2)

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = \rho \frac{\partial^2}{\partial t^2} y_i \tag{4.2}$$

$$[\tau_{ij} n_i] = 0; \quad r \in B.$$

Introducing potentials similar to those previously defined in (3.6), we can transform (4.2) to the equivalent set of wave equations

$$\nabla^2 \chi_\alpha - \frac{1}{v_\alpha^2} \frac{\partial^2}{\partial t^2} \chi_\alpha = -4\pi q_\alpha; \quad \alpha = 1, 2, 3, 4 \tag{4.3}$$

where

$$\chi_k = \tfrac{1}{2}\delta_{knm} \frac{\partial y_m}{\partial x_n}, \quad \chi_4 = \frac{\partial y_l}{\partial x_l}$$

$$v_\alpha = (v_s, v_s, v_s, v_p)$$

$$q_\alpha = \frac{1}{4\pi} \begin{pmatrix} \dfrac{1}{2v_s^2} (\nabla \cdot \mathbf{F}) \cdot \hat{\mathbf{e}}_k \\[2ex] \dfrac{1}{v_p^2} (\nabla \cdot \mathbf{F}) \end{pmatrix}, \quad k = 1, 2, 3$$

with $\hat{\mathbf{e}}_k$ the unit vector in the x_k direction. Note that the summation convention is applied only to Roman indices, so the repeated Greek index α in (4.3) does not imply summation.

As was noted earlier, the analogous 'static' potentials χ_α^* obtained using the stress differences $\sigma_{ij}^{(0)} - \sigma_{ij}^{(1)}$ were harmonic (equation (3.7)). In addition, the χ_α^* represent the changes in the equilibrium values of the dynamic potentials $\chi_\alpha(\mathbf{r}, t)$ just introduced. To see this imagine a hole punched (instantaneously) in the stressed medium, then because of the new boundary condition imposed by the presence of the hole, the medium must relax to a new equilibrium state, defined as $\sigma_{ij}^{(1)}$ in terms of the stress. This change in the equilibrium field is also expressed by χ_α^*, a function of the change in the stress state $\sigma_{ij}^{(0)} - \sigma_{ij}^{(1)}$. Since the dynamic potentials χ_α must eventually pass to the static limit, this means that the χ_α must start at χ_α^* and after a long time approach zero. Clearly χ_α^* is an initial value for χ_α, defined at the instant the hole is punched, and the dynamical problem can be treated as a classical initial value problem. Now if we wish to consider a slightly more complicated problem, we may imagine a succession of such punching operations, separated by some infinitesimal increment in time, $\delta\tau$. Each such operation enlarges the hole by an amount δR in radius and the ratio $\delta R/\delta\tau$ corresponds to a rupture velocity v_R. Each punching operation defines an initial value problem, where the initial value χ_α^* of the potential now depends on the difference between the 'static' equilibrium state prior to enlargement of the hole with that defined for the enlarged hole. In this way the successive incremental changes in the initial values appropriate at each stage of the hole boundary growth are coupled together; yet, since the relevant dynamical equations are linear, we see that the whole process is a superposition of initial value problems, to be summed for the total effect. We note that each operation of punching defines a new initial value problem and that the relaxation is not instantaneous but controlled by the dynamical equations of equilibrium and so obey the usual causality conditions.

A formulation of this dynamical phenomenon follows quite naturally from Green's function solutions to (4.3). The physical processes in this application replacing the 'punching operations' employed in the previous discussion, are fine scale fracturing of the material by the shock wave in the near source zone, perhaps followed by radial cracking around the inner crushed zone due to overpressure in the cavity.

The form of the integral expressions for the dynamical relaxation process does not depend on the details of how the zone of weakness is formed so that these basic results will be given first, and then particular processes and geometries will be considered as special results appropriate to an explosion in a prestressed medium.

The solutions to equations (4.3) are given by the usual Green's function solutions as (e.g. Morse & Feshbach, Vol. 1, 1953)

$$\chi_\alpha(\mathbf{r}, t) = \int_0^{t^+} dt_0 \int_V G_\alpha(\mathbf{r}, t;\ \mathbf{r}_0, t_0)\, q_\alpha(\mathbf{r}_0, t_0)\, d\mathbf{r}_0$$

$$+\frac{1}{4\pi}\int_0^{t^+} dt_0 \int_S \{G_\alpha[\nabla_0\,\chi_\alpha] - [\chi_\alpha]\,\nabla_0\, G_\alpha\}\,.d\mathbf{S}_0$$

$$+\frac{1}{4\pi v_\alpha^2}\int_0^{t^+} dt_0 \int_V \frac{\partial}{\partial t_0}\left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0} - G_\alpha \frac{\partial \chi_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 \qquad (4.4)$$

where $G_\alpha(\mathbf{r}, t;\ r_0, t_0)$ is given by

$$\nabla_0^2\, G_\alpha(\mathbf{r}, t;\ \mathbf{r}_0, t_0) - \frac{1}{v_\alpha^2}\frac{\partial^2}{\partial t_0^2} G_\alpha(\mathbf{r}, t;\ \mathbf{r}_0, t_0) = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)$$

with $G_\alpha = \dfrac{\partial G_\alpha}{\partial t_0} = 0$ for $t < t_0$; for all $\alpha = 1, 2, 3, 4$.

The variables $\mathbf{r}_0$ and t_0 are source spatial and time variables, $\mathbf{r}$ and t the observer's space-time variables and $t^+ = t+\varepsilon$, $\varepsilon > 0$, is used to avoid the singular points of the Green's function. The bracket notation (e.g. $[\nabla_0\,\chi_\alpha]$) is used in the surface integral to express the change in the quantity within the bracket upon crossing S. Thus if S is an exterior boundary to the medium then we define all field quantities to be zero outside the medium and the bracket is just equal to the value assumed on S, approached from inside the medium. On the other hand if S is an interior boundary of material discontinuity, the bracketed quantity is equal to the jump in that quantity across S. If G_α is not chosen to be continuous across boundaries then the same jump conditions apply to it.

The three integral terms in (4.4) have clear physical interpretations, the first is the usual source effect due to an 'externally' applied energy source represented by the q_α. The second corresponds to reflections, refractions, etc. from material boundaries. The third term is less familiar but its physical meaning is easily deduced from a consideration of special cases. Thus, if we were to assume all the functions in the integrand to be continuous functions of time in the range of integration then

$$\int_0^{t^+} dt_0 \int_V \frac{\partial}{\partial t_0}\left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0} - G_\alpha \frac{\partial \chi_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 = \int_V \left[\chi_\alpha \frac{\partial G_\alpha}{\partial t_0} - G_\alpha \frac{\partial \chi_\alpha}{\partial t_0}\right]_0^{t^+} d\mathbf{r}_0 \equiv 0$$

since G_α and $\partial G_\alpha/\partial t_0$ vanish for $t_0 > t$, by construction, while χ_α and $\partial \chi_\alpha/\partial t_0$ vanish for $t_0 < 0$ and because of continuity, also at $t_0 = 0$. On the other hand if χ_α and/or

$\partial\chi_\alpha/\partial t$ may be discontinuous in t_0, say for example that χ_α has an initial value at $t_0 = 0$, then the integral has value and in this instance

$$\int_0^{t+} dt_0 \int_V \frac{\partial}{\partial t_0}\left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0} - G_\alpha \frac{\partial \chi_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 = -\int_V \chi_\alpha(\mathbf{r}_0, 0)\left(\frac{\partial G_\alpha}{\partial t_0}\right)_{t_0=0} d\mathbf{r}_0.$$

If we consider the conceptual experiment involving a single punching operation in a prestressed medium where the time at which the hole is punched is given by t_0, which can be taken to be $t_0 = 0$, and the position of the hole given (implicitly) by $\mathbf{r}_0$ through the dependence of the initial χ_α on $\mathbf{r}_0$, then the expression above describes this initial value problem, where in addition

$$\chi_\alpha(\mathbf{r}_0, 0) = \chi_\alpha^*(\mathbf{r}_0).$$

Here χ_α^* is the potential describing the (total) change in the equilibrium field and is a harmonic function, as was already noted. Hence, if the medium is prestressed then χ_α will have discontinuities related to the changes in the stress field arising from changes in the boundaries of the medium, however these changes are effected. The final integral in (4.4) will therefore describe the dynamical effects of rupture phenomenon in a prestressed medium.

Since the integral terms in (4.4) are additive, we can consider the final term separately. The stress wave radiation from the explosion itself can be accounted for by using either the first or second integral to represent the pressure pulse in the cavity boundary, and added later. Thus we can write the solutions to (4.4) as

$$\chi_\alpha(\mathbf{r}, t) = \chi_\alpha^{(0)} + \chi_\alpha^{(1)} + \chi_\alpha^{(2)} \tag{4.5}$$

where $\chi_\alpha^{(1)}$ and $\chi_\alpha^{(2)}$ represent the two kinds of source effects, $\chi_\alpha^{(1)}$ representing the stress relaxation effects, $\chi_\alpha^{(2)}$ the explosive generated radiation. The solution $\chi_\alpha^{(0)}$ represents the surface integrals in (4.4) or equivalently a general solution to the homogeneous wave equation for χ_α with the arbitrary coefficients of this solution adjusted so that the sum of solutions in (4.5) satisfies the boundary conditions on the surfaces S. We can therefore consider $\chi_\alpha^{(1)}$ separately in an infinite space, and so can use an infinite space Green's function:

$$G_\alpha(\mathbf{r}, t;\ \mathbf{r}_0, t_0) = \frac{\delta(r^*/v_\alpha + t_0 - t)}{r^*}; \quad r^* = |\mathbf{r} - \mathbf{r}_0|. \tag{4.6}$$

The effects of boundaries in so far as they give rise to interference effects, including scattering effects on the growing rupture boundary around the explosion, must then be accounted for by proper choice of the arbitrary coefficients in $\chi_\alpha^{(0)}$. The remainder of this discussion can therefore be directed to the evaluation of $\chi_\alpha^{(1)}$, with G_α as given in (4.6). The effects of the inhomogeneous, layered, medium need not be taken up here.

Thus we require the value of

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = \frac{1}{4\pi v_\alpha^2} \int_0^{t+} dt_0 \int_V \frac{\partial}{\partial t_0}\left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0} - G_\alpha \frac{\partial \chi_\alpha}{\partial t_0}\right\} d\mathbf{r}_0. \tag{4.7}$$

In the simplest case we may think of a rupture zone created at a rate controlled by the shock wave velocity, with the rupture velocity v_R greater than the compressional velocity in the medium and assumed constant. In this case the fracture zone, taken as roughly spherical of final radius R_0, grows without relaxation occurring beyond the rupture front. Therefore if $t_0 = 0$ is the time when an explosion occurs, then at $t_0 = R_0/v_R$ the rupture zone is complete and relaxation occurs thereafter.

The initial value of the potential χ_α is given by

$$\chi_\alpha^*(\mathbf{r}_0) = \left(\frac{1}{r_0}\right)^3 \sum_{m=0}^{2} \{a_{2m}^{(\alpha)} \cos m\phi_0 + b_{2m}^{(\alpha)} \sin m\phi_0\} P_2^m(\cos\theta_0). \tag{4.8}$$

The result can be verified by using (3.18) and the definitions of (3.6) for the χ_α^*. The coefficients $a_{2m}^{(\alpha)}$ and $b_{2m}^{(\alpha)}$ are found to be

$$(a_{2m}^{(\alpha)}) = \frac{5[(1-\sigma)-\delta_{\alpha 4}\sigma]}{\mu(7-5\sigma)} R^3 \begin{pmatrix} 3\sigma_{23}^{(0)} & \sigma_{12}^{(0)} & \sigma_{23}^{(0)}/2 \\ -3\sigma_{13}^{(0)} & 0 & \sigma_{13}^{(0)}/2 \\ 0 & -\sigma_{23}^{(0)} & -\sigma_{12}^{(0)} \\ 0 & \sigma_{13}^{(0)} & 0 \end{pmatrix} \tag{4.9}$$

$$(b_{2m}^{(\alpha)}) = \frac{5[(1-\sigma)-\delta_{\alpha 4}\sigma]}{\mu(7-5\sigma)} R^3 \begin{pmatrix} 0 & 0 & -\sigma_{13}^{(0)}/2 \\ 0 & -\sigma_{12}^{(0)} & \sigma_{23}^{(0)}/2 \\ 0 & \sigma_{13}^{(0)} & 0 \\ 0 & \sigma_{23}^{(0)} & \sigma_{12}^{(0)}/2 \end{pmatrix} \tag{4.10}$$

with $m = 0, 1, 2$ and $\delta_{\alpha 4} = 1$ for $\alpha = 4$ and zero otherwise. Clearly we can define the equilibrium potential χ_α^* for a rupture of this type at any time t_0 by observing that $R = v_R t_0$ is the radius of the fracture zone. In general, then the coefficients in (4.9) and (4.10) are function of t_0.

In the simple case at hand we observe that the initial value problem is defined at $t_0 = R_0/V_R$, the time at which the rupture zone is completely formed, and that the medium outside the fracture zone is initially at rest, but initially displaced from equilibrium. Therefore $\chi_x = \chi_\alpha^*$ and $\partial\chi_\alpha/\partial t_0 = 0$ at $t_0 = R_0/v_R$, so from (4.7)

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = -\frac{1}{4\pi v_\alpha^2} \int_V \chi_\alpha^* \left(\frac{\partial G_\alpha}{\partial t_0}\right)_{t_0 = R_0/v_R} d\mathbf{r}_0 \tag{4.11}$$

where χ_α^* is given by (4.8) and the coefficients $a_{2m}^{(\alpha)}$ and $b_{2m}^{(\alpha)}$ are evaluated at $R = R_0$. This is the result of a 'classical' kind of initial value problem. We will calculate the field $\chi_\alpha^{(1)}$ from (4.11) in the next section. The integration is over the volume V outside the rupture zone.

Suppose we consider the case in which the rupture grows at a rate v_R less than v_S, the shear velocity. This is typical of a natural fracture rate as opposed to a fracture front drive by shock. In this case the medium will begin to relax before the fracture zone is at its final state, and in fact will relax continuously throughout rupture growth. The situation is analogous to the sequence of punching operations discussed earlier. Thus we can conceive of a sequence of events like that described by (4.11), each separated by a short interval of time $\delta\tau$. After each event occurring at time τ_k, $k = 1, 2, \ldots, N$; with $\tau_{k+1} = \tau_k + \delta\tau$, we have a new equilibrium field χ_α^* defined, since the boundary dimension has changed. Thus χ_α will have discontinuities at $t_0 = \tau_k$. However, as in the previous simple case, $\partial\chi_\alpha/\partial t_0$ is zero initially and continuous throughout the sequence of events. Thus, since we cannot integrate (4.7) without accounting for the discontinuous nature of the integrand, we break the integral into a sum of integrals over time intervals for which χ_α is continuous and have

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = \left(\frac{1}{4\pi v_\alpha^2}\right) \lim_{\varepsilon\to 0} \left[\int_0^{\tau_1-\varepsilon} dt_0 \int_{V_1} \frac{\partial}{\partial t_0} \left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 + \int_{\tau_1+\varepsilon}^{\tau_2-\varepsilon} dt_0 \int_{V_2} \frac{\partial}{\partial t_0} \left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 + \ldots + \int_{\tau_N+\varepsilon}^{t+} dt_0 \int_{V_N} \frac{\partial}{\partial t_0} \left\{\chi_\alpha \frac{\partial G_\alpha}{\partial t_0}\right\} d\mathbf{r}_0 \right]$$

where the integrals over the factor involving $\partial\chi_\alpha/\partial t_0$ vanish due to continuity of the integrand. Now evaluating each of the remaining integrals over t_0 gives

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = \left(\frac{1}{4\pi v_\alpha^2}\right) \sum_{k=1}^{N} \int_{V_k} \delta\chi_\alpha(\tau_k) \left(\frac{\partial G_\alpha}{\partial t_0}\right)_{\tau_k} d\mathbf{r}_0 \tag{4.12}$$

where the $V_k = V(\tau_k)$ denote changes in the integration volume as the failure zone enlarges and

$$\delta\chi_\alpha(\tau_k) = \lim_{\varepsilon\to 0} [\chi_\alpha(\tau_k-\varepsilon) - \chi_\alpha(\tau_k+\varepsilon)].$$

But in the limit as $\varepsilon \to 0$, the change, $\delta\chi_\alpha$, is equivalent by definition to the change in the equilibrium field due to the increment of boundary growth occurring at τ_k. Thus

$$\delta\chi_\alpha(\tau_k) = \lim_{\varepsilon\to 0} [\chi_\alpha^*(\tau_k-\varepsilon) - \chi_\alpha^*(\tau_k+\varepsilon)] = \delta\chi_\alpha^*(\tau_k).$$

If we multiply and divide by $\delta\tau$ inside the sum, and take the limit as $\delta\tau$ becomes small and N correspondingly large we get, using the definition of an integral

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = -\frac{1}{4\pi v_\alpha^2} \int_0^{t+} S(\tau_0 - t_0)\, dt_0 \int_{V(t_0)} \frac{\partial\chi_\alpha^*}{\partial t_0} \left(\frac{\partial G_\alpha}{\partial t_0}\right) d\mathbf{r}_0 \tag{4.13}$$

and this result is appropriate to continuous rupture growth at a rate $v_R < v_\alpha$. Here $S(\tau_0 - t_0)$ is a step function, with $\tau_0 \equiv R_0/v_R$, which is unity for $t_0 < \tau_0$ and zero otherwise. It merely serves to delineate the time interval of rupturing. Note that if we were to consider $v_P > v_R > v_S$, then $\chi_4^{(1)}$ would be given by (4.12) while $\chi_k^{(1)}$, $k = 1, 2, 3$, would be given by (4.11).

These results provide quantitative verification of previous expectations. In particular, equation (4.12) shows that the dynamical effects of each incremental 'event' superpose and that the individual fields due to these 'events' obey independent causality relations. That is, stress changes occurring at a point $\mathbf{r}$ and associated with the k-th increment of boundary growth can be considered to begin at a time $|\mathbf{r}/v_\alpha + \tau_k|$. This follows from the form of G_α in (4.6) and from the fact that the Green's function time derivative is evaluated at the source time τ_k. Of course the relaxation in response to any one of the incremental changes in boundary requires infinite time to be completed. On the other hand we see that there is a strong coupling between the incremental 'events' though the factor χ_α^*. Thus the relaxation process which begins at time $|\mathbf{r}/v_\alpha + \tau_k|$ depends on the previous state of equilibrium defined by the $(k-1)$ 'event'.

Equation (4.13) expresses the dynamical response of the medium to continuous boundary growth in a prestressed medium and holds in the general circumstances prevailing for earthquakes, as well as for explosions. The former are considered in this context by Archambeau & Minster (1972).

5. Radiation from explosive sources

The evaluation of the integral solutions is straightforward for explosive sources because we can reasonably assume that the progressive rupture is spherical and centred at one point at all times. The largest uncertainty in the assumptions we will have to make is in the nature of the time history of the rupture process. For this reason several conditions on the rupture velocity will be assumed to provide some degree of generality in the solution set. Taken as a group, the assumptions to be employed should provide reasonable bounds to the conditions likely to prevail for explosions.

The general assumptions common to all the cases to be considered are that the stress and elastic properties are uniform in the source region and pure shear with stress relaxation a local phenomenon; that the rupture velocity is piecewise constant over the time interval of rupturing and that the rupture zone maintains spherical symmetry about a fixed point throughout the failure process. None of these assumptions is necessarily required for the evaluation of the integral solutions, but they are reasonable simplifications in view of observations and the previous discussion involving the tectonic energy release for explosions.

Collecting previous results and using the infinite space Green's function solution, we have, for $v_R < v_\alpha$

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = -\frac{1}{4\pi v_\alpha^2} \int_0^{t^+} S(\tau_0 - t_0)\, dt_0 \int_{V(t_0)} \frac{\partial \chi_\alpha^*}{\partial t_0} \left(\frac{\delta_1(r^*/v_\alpha + t_0 - t)}{r^*} \right) d\mathbf{r}_0$$

$$\chi_\alpha^*(\mathbf{r}_0) = \left(\frac{1}{r_0} \right)^3 \sum_{m=0}^{2} \{a_{2m}^{(\alpha)}(t_0) \cos m\phi_0 + b_{2m}^{(\alpha)}(t_0) \sin m\phi\}\, P_2^m(\cos\theta_0)$$

with the $a_{2m}^{(\alpha)}$ and $b_{2m}^{(\alpha)}$ given by (4.9) and (4.10). Here

$$\delta_1(r^*/v_\alpha + t_0 - t) = \frac{\partial}{\partial t_0} \delta(r^*/v_\alpha + t_0 - t) = -\frac{\partial}{\partial t} \delta(r^*/v_\alpha + t_0 - t).$$

We will consider the far field, so that the observers time t will be taken larger than $\tau_0 = R_0/v_R$. In this case, the solution has the form, using (4.13):

$$\chi_\alpha^{(1)}(\mathbf{r}, t) = -\frac{1}{4\pi v_\alpha^2} \int_0^{\tau_0} dt_0 \int_{V(t_0)} \frac{\partial \chi_\alpha^*}{\partial t_0} \left(\frac{\delta_1(r^*/v_\alpha + t_0 - t)}{r^*} \right) d\mathbf{r}_0. \tag{5.1}$$

It is convenient to take Fourier transforms with respect to t, and to evaluate the field in the frequency domain. Taking this transform gives, with $k_\alpha = \omega/v_\alpha$:

$$\tilde{\chi}_\alpha^{(1)}(\mathbf{r}, \omega) = \frac{i\omega}{4\pi v_\alpha^2} \int_0^{\tau_0} \exp(-i\omega t_0) \int_{V(t_0)} \frac{\partial \chi_\alpha^*}{\partial t_0} \left[\frac{\exp(-ik_\alpha r^*)}{r^*} \right] d\mathbf{r}_0\, dt_0.$$

Now introducing the usual spherical wave expansion (e.g. Morse & Feshbach 1953)

$$\frac{1}{r^*} \exp(-ik_\alpha r^*) = -ik_\alpha \sum_{l=0} (2l+1)\, P_l(\cos\gamma)\, j_l(k_\alpha r_0)\, h_l^{(2)}(k_\alpha r)$$

for $r > R_s \geqslant r_0$, where we use R_s as the effective source volume, and integrating over the angular variables, after introducing the harmonic expansion, we have

$$\tilde{\chi}_\alpha^{(1)}(\mathbf{r}, \omega) = h_2^{(2)}(k_\alpha r) \sum_{m=0}^{2} [A_{2m}^{(\alpha)}(\omega) \cos m\phi + B_{2m}^{(\alpha)}(\omega) \sin m\phi] \, P_2^m(\cos\theta) \tag{5.2}$$

$$\begin{pmatrix} A_{2m}^{(\alpha)}(\omega) \\ B_{2m}^{(\alpha)}(\omega) \end{pmatrix} = \frac{k_\alpha^2}{v_\alpha} \int_0^{\tau_0} \frac{\partial}{\partial t_0} \begin{pmatrix} a_{2m}^{(\alpha)}(t_0) \\ b_{2m}^{(\alpha)}(t_0) \end{pmatrix} \times \exp(-i\omega t_0) \int_{v_R t_0}^{R_s} \left(\frac{1}{r_0}\right) j_2(k_\alpha r_0) \, dr_0 \, dt_0. \tag{5.3}$$

Thus the solution for the transformed potential $\tilde{\chi}_\alpha^{(1)}$ has a simple quadrupole form and the coefficients have the frequency dependence specified by (5.3).

The result expressed by (5.3) constrains the relaxation effects to the radial region $(v_R t_0, R_s)$ when $v_R < v_\alpha$. That is, it is assumed that the medium behaves elastically in the region outside the rupture front at any given time. An alternative assumption which is equally reasonable would be to take the elastic (relaxation) zone to be fixed within the region (R_0, R_s). This would imply that the strain energy within the final rupture region $(0, R_0)$ is converted to heat by non-elastic processes and does not contribute to the radiation field at any time during the growth of the rupture. Under this assumption the expressions for the radiation coefficients are simpler since we would then replace the lower limit $v_R t_0$ in (5.3) by R_0, to give

$$\begin{pmatrix} A_{2m}^{(\alpha)}(\omega) \\ B_{2m}^{(\alpha)}(\omega) \end{pmatrix} = \frac{k_\alpha^2}{v_\alpha} \int_0^{\tau_0} \frac{\partial}{\partial t_0} \begin{pmatrix} a_{2m}^{(\alpha)}(t_0) \\ b_{2m}^{(\alpha)}(t_0) \end{pmatrix} \exp(-i\omega t_0) \, dt_0 \int_{R_0}^{R_s} \frac{1}{r_0} j_2(k_\alpha r_0) \, dr_0.$$

For the case $v_R > v_\alpha$ we have from (4.11), after taking Fourier transforms

$$\tilde{\chi}_\alpha^{(1)}(\mathbf{r}, \omega) = \frac{i\omega}{4\pi v_\alpha^2} \exp[-ik_R R_0] \int_V \chi_\alpha^* \left(\frac{\exp(-ik_\alpha r^*)}{r^*} \right) d\mathbf{r}$$

where $k_R = \omega/v_R$. Introducing the spherical wave expansion and integrating over the angular variables then gives

$$\tilde{\chi}_\alpha^{(1)}(\mathbf{r}, \omega) = h_2^{(2)}(k_\alpha r) \sum_{m=0}^{2} [A_{2m}^{(\alpha)}(\omega) \cos m\phi + B_{2m}^{(\alpha)}(\omega) \sin m\phi] \, P_2^m(\cos\theta) \tag{5.4}$$

with

$$\begin{pmatrix} A_{2m}^{(\alpha)}(\omega) \\ B_{2m}^{(\alpha)}(\omega) \end{pmatrix} = \frac{k_\alpha^2}{v_\alpha} \exp[-ik_R R_0] \int_{R_0}^{R_s} \frac{1}{r_0} j_2(k_\alpha r_0) \, dr_0 \begin{pmatrix} a_{2m}^{(\alpha)} \\ b_{2m}^{(\alpha)} \end{pmatrix}. \tag{5.5}$$

The spatial integrals appearing in (5.3) and (5.5) are known (see Erdelyi *et al.* 1954).

Therefore we have, in general,

$$\tilde{\chi}_\alpha^{(1)}(\mathbf{r}, \omega) = h_2^{(2)}(k_\alpha r) \sum_{m=0}^{2} [A_{2m}^{(\alpha)} \cos m\phi + B_{2m}^{(\alpha)}(\omega) \sin m\phi] \, P_2^m(\cos\theta) \tag{5.6a}$$

where, for $v_R \geqslant v_\alpha$

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)(k_\alpha R_0)^2 \exp\left[-ik_R R_0\right]$$

$$\times\left\{\frac{j_1(k_\alpha R_0)}{k_\alpha R_0}-\frac{j_1(k_\alpha R_s)}{k_\alpha R_s}\right\}\cdot\begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix} \quad (5.6\text{b})$$

and for $v_R < v_\alpha$

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)}\left(\frac{k_\alpha^2}{v_\alpha}\right)\cdot\begin{pmatrix} \Sigma_{\alpha m}^{(\alpha)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix}\int_0^{R_0/v_R}\frac{\partial}{\partial t_0}\left[(v_R t_0)^3\right]$$

$$\times\left\{\frac{j_1(k_\alpha v_R t_0)}{k_\alpha v_R t_0}-\frac{j_1(k_\alpha R_s)}{k_\alpha R_s}\right\}\exp\left(-i\omega t_0\right)dt_0 \quad (5.6\text{c})$$

or, when all the energy within the region $(0, R_0)$ is continuously converted to heat then

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)}\left(\frac{k_\alpha^2}{v_\alpha}\right)\int_0^{R_0/v_R}\frac{\partial}{\partial t_0}\left[(v_R t_0)^3\right]\exp\left(-i\omega t_0\right)dt_0$$

$$\times\left\{\frac{j_1(k_\alpha R_0)}{k_\alpha R_0}-\frac{j_1(k_\alpha R_s)}{k_\alpha R_s}\right\}\begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix}. \quad (5.6\text{d})$$

The symbols $\Sigma_{\alpha m}^{(\alpha)}$ and $\Sigma_{\alpha m}^{(b)}$ denote the (stress) matrices in equations (4.9) and (4.10), respectively.

(1) $v_R \geqslant v_\alpha$

Observing that

$$\frac{j_1(k_\alpha R_0)}{k_\alpha R_0} = \left(\frac{1}{k_\alpha R_0}\right)^2\left[\frac{\sin k_\alpha R_0}{k_\alpha R_0} - \cos k_\alpha R_0\right]$$

we have, in terms of elementary functions, the following solutions and limiting forms:

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_R R_0\right]$$

$$\times\left\{\left(\cos(k_\alpha R_0)-\frac{\sin k_\alpha R_0}{k_\alpha R_0}\right)-\left(\frac{R_0}{R_s}\right)^2\cdot\left(\cos(k_\alpha R_s)-\frac{\sin k_\alpha R_s}{k_\alpha R_s}\right)\right\}$$

$$\times\begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix} \quad (5.7\text{a})$$

which holds in general.

In the long wavelength limit, $k_\alpha R_0 < k_\alpha R_s \ll 1$ we have

$$\cos(k_\alpha R_0)-\frac{\sin k_\alpha R_0}{k_\alpha R_0} \simeq -\frac{(k_\alpha R_0)^2}{3}+\frac{(k_\alpha R_0)^4}{30}$$

so that (5.7a) is, approximately:

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = -\left(\frac{1}{6}\right)\frac{(1-\sigma)-\delta_{\alpha 4}\,\sigma}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_R R_0\right]\left\{1-\left(\frac{R_0}{R_s}\right)^2\right\}$$

$$\times (k_\alpha R_0)^2 (k_\alpha R_s)^2 \begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix}. \qquad (5.7\text{b})$$

If $R_s^2 \gg R_0^2$ as well, then the factor $(R_0/R_s)^2$ can be neglected in (5.7b). In any case we note that the frequency dependence of the solution coefficients is such that

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = 0(\omega^4); \quad \text{for} \quad k_\alpha R_0 < k_\alpha R_s \ll 1.$$

We note that for R_s to be near R_0 means that the effective source volume is small and the spectrum at long periods is reduced by the factor $\{1-(R_0/R_s)^2\}$. If the prestress were highly localized and dropped off rapidly from the source origin for some reason, then we would expect R_s to have a value near the characteristic dimension of such a stress concentration. If the prestress is uniform over a region of radius sufficiently large compared to the fracture zone radius R_0, then we can use the result previously obtained and illustrated in Fig. 8; that is that most of the energy release is from within a radius R_s of from three to four times R_0. In this case $(R_0/R_s)^2 \ll 1$ in the expressions for both the energy release and for the radiation field coefficients. Since we normally expect R_0, the fracture zone radius, to be of the order of 1 km for explosions, it is plausible that a prestress field uniform over a region of radius three to four times this amount would prevail in most instance. Thus we can usually neglect terms of order $(R_0/R_s)^2$ compared to one, in the theory, as well as to ignore any stress variations which may exist at relatively modest distances from the source.

(2) $v_R < v_\alpha$

The evaluation of the integral solution (5.6c) is straightforward but lengthy. Evaluating the elementary integrals appearing in (5.6c), we have the basic result

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left(-ik_R R_0\right)\left\{C(k_R;\ k_\alpha)-\left(\frac{R_0}{R_s}\right)^2\right.$$

$$\left.\times\, {}_1F_1(1;\ 4;\ ik_R R_0)\left[\frac{\sin k_\alpha R_s}{k_\alpha R_s}-\cos k_\alpha R_s\right]\right\}\begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix} \qquad (5.8)$$

with the residual integral

$$C(k_R;\ k_\alpha) = \exp\left(ik_R R_0\right)\int_0^1 \left[\frac{\sin k_\alpha R_0 t'}{k_\alpha R_0 t'} - \cos k_\alpha R_0 t'\right].\exp\left[-ik_R R_0 t'\right] dt'.$$

This integral can be evaluated in a number of analytic forms; however, all of them are complicated and are difficult to evaluate numerically over the entire frequency range of interest. The best general approach is to evaluate it numerically as a Fourier transform. However, some features of the analytic character of this factor can be determined from the series form

$$C(k_R;\ k_\alpha) = 3\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2n}{2n+1}\right)\frac{(k_\alpha R_0)^{2n}}{\Gamma(2n+2)}\,{}_1F_1(1;\ 2n+2;\ ik_R R_0).$$

This is also convenient for computation when both $k_R R_0$ and $k_\alpha R_0$ are much smaller than unity. The ${}_1F_1(a;\ b;\ \zeta)$ are confluent hypergeometric functions, and the important specific functions appearing here can be written in terms of trigonometric functions as,

$$ {}_1F_1(1;\ 2;\ -i\zeta) = \exp\left(-\frac{i}{2}\zeta\right)\frac{\sin(\zeta/2)}{(\zeta/2)} $$

$$ {}_1F_1(1;\ 4;\ i\zeta) = -\frac{3}{i\zeta}\left[1+\frac{2}{i\zeta}\left\{\frac{1-\exp(i\zeta)}{i\zeta}+1\right\}\right] $$

$$ \lim_{\zeta\to\infty} {}_1F_1(1;\ n;\ i\zeta) = 0, \quad \lim_{\zeta\to 0} {}_1F_1(1;\ n;\ i\zeta) = 1, \quad n = 2, 4. $$

Thus (5.8), while rather lengthy, can easily be evaluated numerically. We note in passing that if we allow $v_R \to 0$, then all the $A_{2m}{}^{(\alpha)}$ and $B_{2m}{}^{(\alpha)}$ vanish in view of the properties of the functions defined above. This of course is the expected result.

In the long wavelength limit, such that $k_\alpha R_0 < k_\alpha R_s \ll 1$, then using the series for $C(k_R;\ k_\alpha)$ to second order, we have

$$ \begin{pmatrix} A_{2m}{}^{(\alpha)} \\ B_{2m}{}^{(\alpha)} \end{pmatrix} = -\left(\frac{1}{6}\right)\frac{(1-\sigma)-\delta_{\alpha 4}\sigma}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_R R_0\right] $$
$$ \times\left\{{}_1F_1(1;\ 4;\ ik_R R_0)-\frac{3}{5}\left(\frac{R_0}{R_s}\right)^2 {}_1F_1(1;\ 6;\ ik_R R_0)\right\}(k_\alpha R_0)^2(k_\alpha R_s)^2\begin{pmatrix} \Sigma_{\alpha m}{}^{(a)} \\ \Sigma_{\alpha m}{}^{(b)} \end{pmatrix}. \tag{5.9a} $$

If we are dealing with a reasonably uniform prestress field in the vicinity of the fracture zone, then $(R_0/R_s)^2 \ll 1$ and we have the approximate result

$$ \begin{pmatrix} A_{2m}{}^{(\alpha)} \\ B_{2m}{}^{(\alpha)} \end{pmatrix} = -\left(\frac{1}{6}\right)\frac{(1-\sigma)-\delta_{\alpha 4}\sigma}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_R R_0\right]{}_1F_1(1;\ 4;\ ik_R R_0) $$
$$ \times(k_\alpha R_0)^2(k_\alpha R_s)^2\begin{pmatrix} \Sigma_{\alpha m}{}^{(a)} \\ \Sigma_{\alpha m}{}^{(b)} \end{pmatrix} \tag{5.9b} $$

On the other hand, if we take $k_R R_0 \ll 1$ in (5.9a) without assuming $(R_0/R_s)^2 \ll 1$, we have:

$$ \begin{pmatrix} A_{2m}{}^{(\alpha)} \\ B_{2m}{}^{(\alpha)} \end{pmatrix} = -\left(\frac{1}{6}\right)\frac{(1-\sigma)-\delta_{\alpha 4}\sigma}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_R R_0\right]\left\{1-3/5\left(\frac{R_0}{R_s}\right)^2\right\} $$
$$ \times(k_\alpha R_0)^2(k_\alpha R_s)^2\begin{pmatrix} \Sigma_{\alpha m}{}^{(a)} \\ \Sigma_{\alpha m}{}^{(b)} \end{pmatrix}. \tag{5.9c} $$

It is interesting to compare this result with the corresponding limit case for $v_R > v_\alpha$. In particular, the difference between these cases is obtained by comparing (5.7a) and (5.9c) and we note that they differ only in the factor involving $(R_0/R_s)^2$. When $R_0/R_s \to 1$ in (5.7b), then the field vanishes as it should since in this case the region of relaxation which contributes to the radiation field shrinks to zero inasmuch as we *a priori* exclude contributions from the region $|\mathbf{r}| < R_0$. However, for the model for which (5.9c) applies we have only excluded contributions from behind the expanding rupture front, which moves at a rate v_R less than v_α. Hence we expect some contribution from within the region $|\mathbf{r}| < R_0$ and therefore a non-vanishing result when $(R_0/R_s) \to 1$. We see from (5.9c) that we obtain a limit value which is indeed non-zero and is $\frac{2}{5}$ of the value we get when $(R_0/R_s) \to 0$, the limit for uniform prestress. Thus at least a part of the energy stored in the region $|\mathbf{r}| < R_0$ corresponding to the final zone of fracture is released during slow rupture.

In the long wave limit such that $k_R R_0 \ll 1$, $k_\alpha R_0 < k_\alpha R_s \ll 1$ and when $(R_0/R_s)^2 \ll 1$ in addition, then (5.9c) reduces to the form which is identical with the limiting solution for the $v_R > v_\alpha$ when these same limiting conditions are applied. This is as we would expect, since for waves of period long compared to the finite time of rupture formation (R_0/v_R), it should not matter to first order whether v_R is somewhat larger or smaller than the intrinsic wave velocity v_α, and the results for the two formulations should agree.

Finally, if we consider the model expressed by (5.6d) where we assumed that the strain energy initially stored within the final rupture region $(0, R_0)$ is converted to heat even when $v_R < v_\alpha$, then we have

$$\begin{pmatrix} A_{2m}{}^{(\alpha)} \\ B_{2m}{}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)} \left(\frac{R_0}{v_\alpha}\right) \exp\,(-ik_R R_0)\,{}_1F_1(1;\ 4;\ ik_R R_0)$$

$$\times \left\{\left(\frac{\sin k_\alpha R_0}{k_\alpha R_0} - \cos k_\alpha R_0\right) - \left(\frac{R_0}{R_s}\right)^2 \left(\frac{\sin k_\alpha R_s}{k_\alpha R_s} - \cos k_\alpha R_s\right)\right\} \begin{pmatrix} \Sigma_{\alpha m}{}^{(a)} \\ \Sigma_{\alpha m}{}^{(b)} \end{pmatrix}. \quad (5.10)$$

If $k_R R_0$ is small, so that either the rupture velocity is taken to be large or the wave-length of the radiation long, then this result becomes identical with (5.7a). If $k_\alpha R_0 < k_\alpha R_s \ll 1$ with $(R_0/R_s)^2 \ll 1$, then we obtain an approximation identical to that given by (5.9b).

Additional solutions based on more complicated rupture histories can be envisioned which have relevance. In particular, the discussion accompanying Fig. 7 (Section 3) suggests that we might wish to evaluate the case in which the rupture velocity v_R is greater than the intrinsic elastic velocity v_α out to a radius R_c, after which failure or fine scale cracking occurs out to a final radius R_0 at a rate less than v_α, due to cavity overpressure combined with the non-hydrostatic stresses associated with the prestressed condition of the medium. This model can be analytically specified by taking the rupture velocity to be $v_{R_1} \geqslant v_\alpha$ for a time R_c/v_{R_1}, and then equal to $v_{R_2} < v_\alpha$ during the time interval R_c/v_{R_1} to $R_c/v_{R_1} + (R_0 - R_c)/v_{R_2}$ with v_{R_1} and v_{R_2} constants. This is equivalent to combining previous models treated and we have, after assuming that the strain energy stored in the region R_c to R_0 is dissipated in the non-linear processes of fracture and flow

$$\begin{pmatrix} A_{2m}{}^{(\alpha)} \\ B_{2m}{}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\,\sigma]}{\mu(7-5\sigma)} \left(\frac{R_c}{v_\alpha}\right)^3 \exp\,[-ik_{R_1} R_c]\,\omega^2 \left\{1 + 3\left(\frac{v_{R_2}}{R_c}\right)^3 \exp\,[ik_{R_2} R_c]\right.$$

$$\left.\times \int_{R_c/v_{R_2}}^{R_0/v_{R_2}} t^2 \exp\,(-i\omega t)\, dt\right\} \int_{R_0}^{R_s} \frac{1}{r_0}\, j_2(k_\alpha r_0)\, dr_0 \begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix}.$$

The first term in the brackets (i.e. unity) is associated with the first stage failure driven at the shock speed v_{R_1} and the second term with failure due to combined cavity pressure and tectonic stress overloading proceeding at a rate v_{R_2}, both giving rise to seismic radiation due to relaxation in the region R_0 to R_s. For both stages of failure it is assumed that the strain energy initially stored within the spherical rupture from

0 to R_0 goes into the work of non-elastic deformation and fracture. The integrals in this expression are elementary and the result may be expressed as

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = \frac{-5[(1-\sigma)-\delta_{\alpha 4}\sigma]}{\mu(7-5\sigma)}\left(\frac{R_0}{v_\alpha}\right)\exp\left[-ik_{R_1}R_c\right]\left\{\left(\frac{R_c}{R_0}\right)^3\right.$$

$$+\left[\exp\left[-ik_{R_2}(R_0-R_c)\right]{}_1F_1(1;\ 4;\ ik_{R_2}R_0)-\left(\frac{R_c}{R_0}\right)^3{}_1F_1(1;\ 4;\ ik_{R_2}R_c)\right]\Bigg\}$$

$$\times\left\{\left(\frac{\sin k_\alpha R_0}{k_\alpha R_0}-\cos k_\alpha R_0\right)-\left(\frac{R_0}{R_s}\right)^2\left(\frac{\sin k_\alpha R_s}{k_\alpha R_s}-\cos k_\alpha R_s\right)\right\}\begin{pmatrix} \Sigma_{\alpha m}^{(a)} \\ \Sigma_{\alpha m}^{(b)} \end{pmatrix}. \quad (5.11)$$

The long wave limit and various special cases for this model are obtained quite easily from this expression and are analogous to those obtained for the simpler models. Archambeau & Sammis (1970) give some of these results for this model.

The asymptotic frequency dependence of the multipole coefficients for the various models are easily seen to have the following forms

$$\begin{pmatrix} A_{2m}^{(a)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = 0(\omega^4);\quad \text{when } \omega \ll 1, \text{ for all models.}$$

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = 0(1/\omega);\quad \text{when } \omega \gg 1, \text{ for models with } v_R < v_\alpha.$$

$$\begin{pmatrix} A_{2m}^{(\alpha)} \\ B_{2m}^{(\alpha)} \end{pmatrix} = 0(1);\quad \text{when when } \omega \gg 1, \text{ for models with } v_R > v_\alpha.$$

The displacement field is given by

$$\tilde{\mathbf{u}}(\mathbf{r},\omega) = -1/k_p^2\,\nabla\tilde{\chi}_4 + 2/k_s^2\,\nabla x\,\bar{\chi}$$

where $\bar{\chi} = \tilde{\chi}_k\,\hat{\mathbf{e}}_k$, $k = 1, 2, 3$. A typical component, say $\tilde{u}_r$, is of the form

$$\tilde{u}_r(\mathbf{r},\omega) = -1/k_p^2\frac{\partial\chi_4}{\partial r} - 2/k_s^2\left(\frac{1}{r}\right)\left\{\left(\sin\phi\frac{\partial}{\partial\theta}+\cos\phi\cot\theta\right)\tilde{\chi}_1\right.$$

$$\left.+\left(\sin\phi\cot\theta\frac{\partial}{\partial\phi}-\cos\phi\frac{\partial}{\partial\theta}\right)\tilde{\chi}_2-\frac{\partial}{\partial\phi}\tilde{\chi}_3\right\}$$

and we have, therefore, frequency dependent terms involving $(1/k_\alpha^2)\tilde{\chi}_\alpha$ and $(1/k_\alpha^2)(\partial\chi_\alpha/\partial r)$. Thus the limiting frequency dependence of the displacement spectrum is given by the order relations

$$|\tilde{u}| = 0(1/k_\alpha^2\,A_{2m}^{(\alpha)}(\omega)\,h_2^{(2)}(k_\alpha r))$$

or (depending on whether high or low frequency limits are taken)

$$|\tilde{u}| = 0\left(1/k_\alpha^2\,A_{2m}^{(\alpha)}(\omega)\frac{\partial h_2^{(2)}(k_\alpha r)}{\partial r}\right).$$

Therefore, noting that

$$h_2^{(2)}(k_\alpha r) = \frac{\exp(-ik_\alpha r)}{k_\alpha r}\left[1 - \frac{3}{(k_\alpha r)^2} - i\left(\frac{3}{k_\alpha r}\right)\right]$$

$$\frac{\partial h_2^{(2)}(k_\alpha r)}{\partial r} = -\frac{\exp(-ik_\alpha r)}{r}\left[\left(1 - \frac{9}{(k_\alpha r)^2}\right) - i\left(\frac{4}{k_\alpha r} - \frac{9}{(k_\alpha r)^3}\right)\right]$$

we have for the dominating terms, insofar as frequency dependence is concerned:

(a) $|\tilde{u}| = 0\left(\frac{1}{\omega}\right)$, for $\omega \ll 1$

(b) $|\tilde{u}| = 0\left(\frac{1}{\omega^3}\right)$, for $\omega \gg 1$ and $v_R < v_\alpha$

$|\tilde{u}| = 0\left(\frac{1}{\omega^2}\right)$, for $\omega \gg 1$ and $v_R > v_\alpha$.

It is important to note that the low frequency limit given in (a) corresponds to the 'near field' static offset. That is, the limiting form for $|\tilde{u}|$ at low frequency can be seen to depend on distance as r^{-4}. It is therefore useful to indicate the distance dependence along with the frequency dependence in the low frequency limit. For the various terms contributing at $\omega \ll 1$ we have:

$$|\tilde{u}_1| = 0\left(\frac{1}{\omega}\right), \quad \text{with} \quad |\tilde{u}_1| \sim r^{-4}$$

$$|\tilde{u}_2| = 0(1), \quad \text{with} \quad |\tilde{u}_2| \sim r^{-3}$$

$$|\tilde{u}_3| = 0(\omega), \quad \text{with} \quad |\tilde{u}_3| \sim r^{-2}$$

$$|\tilde{u}_4| = 0(\omega^2), \quad \text{with} \quad |\tilde{u}_4| \sim r^{-1}.$$

The actual spectrum is of course the (weighted) sum of these terms. The term $|\tilde{u}_4|$ is usually called the 'far field' radiation and is taken to be the only significant term at 'sufficiently large' distances. However, sufficiently large distances are those such that $k_\alpha r \gg 1$ and therefore at any finite r this condition will be violated at low low frequencies. Thus for low frequency estimates, all terms should be retained.

In an earlier paper (Archambeau 1968), only the term $|\tilde{u}_2|$ was, in effect, used in a low frequency estimate for $|\tilde{\mathbf{u}}|$ and leads to the statement $|\tilde{\mathbf{u}}| = 0(1)$ for $\omega \ll 1$, which is erroneous as was pointed by A. Linde (1971). The features of the radiation field spectrum due to stress relaxation are illustrated in Fig. 9. The spectra shown were computed from equation (5.10); however, computations using the other models give only minor differences at high frequencies. These typical spectra show the behaviour described by the asymptotic results; in particular at high frequencies the spectrum falls off as $1/\omega^3$ and at very low frequencies the near field terms dominate, with the spectrum varying as $1/\omega$. Note that the near field $1/\omega$ dependence becomes evident at higher frequencies for the spectrum at 5 km than it does for the spectrum at 50 km due to the different dependence on distance for the near and far field terms making up $\tilde{u}(\mathbf{r}, \omega)$.

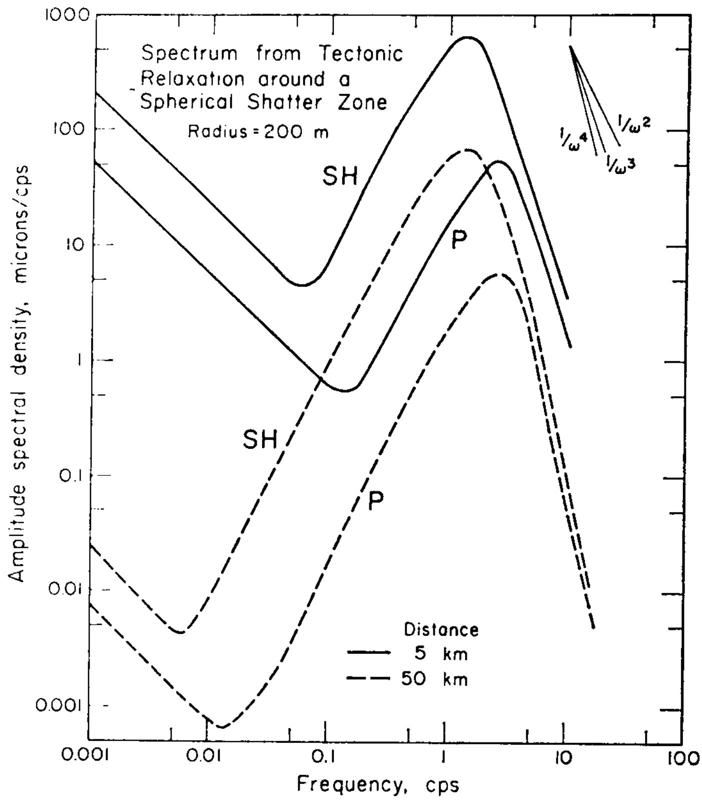

FIG. 9. Displacement spectral density of the direct compressional (P) wave and shear (SH) wave from a spherical relaxation source at two distances from the centre of the failure zone. The relaxation zone radius is taken to be approximately 1 km. The medium elastic properties are appropriate for welded tuff under a uniform shear stress of 30 b.

Fig. 9 also shows a strong peak, at around 1 cps for the source dimensions chosen, with a decrease in the spectra density toward lower frequencies with a slope of roughly ω^2. This variation is appropriate for the far field term $|\tilde{u}_4|$, which dominates in the frequency range just below the peak frequencies at these distances. It is also worth noting that the SH displacements are roughly an order of magnitude larger than the P wave displacements and that the spectral peaks for the shear displacements are shifted to lower frequencies. This is as expected for relaxation in a pure shear prestress field.

So far we have neglected the radiation from the explosion itself, that is the compressional wave from the reduction of the shock wave to an elastic wave. As we have pointed out earlier (equation (4.5)), this contribution $\chi_\alpha^{(2)}$ must be added to $\chi_\alpha^{(1)}$, the field due to relaxation effects. We can always simulate the shock wave conversion by an equivalent source model obtained by choosing a suitable radius R' (an effective cavity radius at the elastic zone boundary) over which a pressure $p_0(t)$ is imposed. Sharpe (1942) has treated this problem in detail and we need only express his results in a slightly modified notation and form. Harkrider (1963) has essentially the form needed and we get, after minor modifications

$$\chi_4^{(2)} = -\frac{\tilde{p}_0(\omega)}{4\mu}(k_p R')^2 \left(\frac{R' \exp[-i(\theta_p - k_p R')]}{\left[\left(1 - \frac{(k_s R')^2}{4}\right)^2 + (k_p R')^2\right]^{\frac{1}{2}}} \right) \times \exp[-ik_p r]/r \quad (5.12a)$$

with

$$\theta_p = \tan^{-1}\left[\frac{k_p R'}{(1-(k_s R')^2/4)}\right]$$

and where

$$\chi_j^{(2)} = 0, \qquad j = 1, 2, 3. \tag{5.12b}$$

We note that $\chi_4^{(2)}$ is the only non-zero potential component so that the field is purely dilatational. The result can of course be expressed as a multipole solution. In particular, since

$$\exp(-ik_p r)/r = -ik_p h_0^{(2)}(k_p r)$$

then

$$\chi_\alpha^{(2)}(\mathbf{r}, \omega) = A_{00}^{(\alpha)}(\omega)\, h_0^{(2)}(k_p r) \tag{5.13}$$

with

$$A_{00}^{(\alpha)}(\omega) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ G(\omega) \end{pmatrix}$$

$$G(\omega) = -\frac{\tilde{p}_0(\omega)}{4\mu}(k_p R')^3 \left(\frac{\exp[-i(\theta_p - k_p R' + \pi/2)]}{\left[\left(1 - \frac{k_s^2 R'^2}{4}\right)^2 + (k_p R')^2\right]^{\frac{1}{2}}} \right)$$

and this result may be directly added to $\chi_4^{(1)}$ to obtain the total field. The displacement field associated with (5.13) is purely radial, and using the relation between the displacement field and the potentials employed in this paper

$$y_i(r, \omega) = \frac{1}{k_p^2}\frac{\partial \chi_4^{(2)}}{\partial x_i} - \frac{2}{k_s^2}\delta_{ijk}\frac{\partial \chi_k^{(2)}}{\partial x_j} \tag{5.14}$$

we have for the components, in spherical co-ordinates

$$y_r(\mathbf{r}, \omega) = \frac{\tilde{p}_0(\omega)}{4\mu} R'^3 \left[\frac{\exp[-i(\theta_p - k_p R')]}{\left[\left(1 - \frac{k_s^2 R'^2}{4}\right)^2 + k_p^2 R'^2\right]^{\frac{1}{2}}} \right] \times (ik_p r + 1)\frac{\exp(-ik_p r)}{r^2} \tag{5.15a}$$

$$y_\theta(\mathbf{r}, \omega) = y_\varphi(\mathbf{r}, \omega) = 0. \tag{5.15b}$$

A similar procedure, using (5.14), yields the displacements due to relaxation effects which add to the results in (5.15). These latter displacements will not be written out here since the results are lengthy. It is worth pointing out, however, that (5.14) is a relation for the Cartesian components of the displacement field.

The pressure $p_0(t)$ and the effective cavity radius R' are unspecified so far. Observations, Toksöz, Ben-Menahem & Harkrider (1964), suggest that an appropriate functional form for $p_0(t)$ is

$$p_0(t) = \begin{cases} P_0\, \tau^{\zeta} \exp(-\eta\tau); & \tau = t - R'/v_R \geqslant 0 \\ 0 & ; \tau < 0 \end{cases}$$

where $t = 0$ is taken as the time of the explosion and where the elastic radius is defined at a time R'/v_R with v_R the rupture or shock velocity.

Also P_0, a constant, is the pressure magnitude dependent on explosive yield. The parameters ζ and η are observed to be bounded by $\frac{1}{3} \leqslant \zeta \leqslant 1$ and $0{\cdot}6 \leqslant \eta \leqslant 1{\cdot}6$. These bounds were obtained by Toksöz *et al.* on the basis of observations of the far field from several explosions. Both ζ and η are (weak) functions of the yield, both approach their upper bounds with increasing yield. For moderately large explosions $\zeta \simeq 1$ and $\eta \simeq 1$ are good approximations. In any case the Fourier transform of $p_0(t)$ is

$$p_0(\omega) = \frac{P_0 \Gamma(\zeta+1)}{(\eta^2+\omega^2)^{(\zeta+1)/2}} \exp\left\{ -i(\zeta+1)\tan^{-1}\frac{\omega}{\eta} - ik_R R' \right\}. \tag{5.16}$$

Note that for $\eta = \zeta = 0$ we have the transform of a step function pressure. For $\zeta = 1$ and $\eta = 1$ it becomes

$$p_0(\omega) = \frac{P_0}{1+\omega^2} \exp\left\{ -\mathrm{i}\tan^{-1}\left[\frac{2\omega}{(1-\omega^2)}\right] - ik_R R' \right\}.$$

Thus $G(\omega)$ appearing in (5.13) is of the general form

$$G(\omega) = \frac{-1}{4\mu} \frac{(k_p R')^3}{(\eta^2+\omega^2)^{(\zeta+1)/2}} P_0 \Gamma(\zeta+1)$$

$$\times \left[\frac{\exp\left\{ -i(\theta_p + (k_R - k_p) R') + (\zeta+1)\tan^{-1}\frac{\omega}{\eta} + \pi/2 \right\}}{\left\{ \left[1 - \frac{k_s^2 R'^2}{4} \right]^2 + [k_p R']^2 \right\}^{\frac{1}{2}}} \right] \tag{5.17}$$

and we now have the field specified in terms of the parameters ζ, η, P_0, R'. Both of the latter parameters are strong functions of yield and medium type whereas the first two are rather insensitive. Toksöz *et al.* (1964), in their Fig. 7, give examples of P_0 and R' (the maximum reversible stress point) for different types of media and explosive yields using Bishop's (1963) data. Haskell (1961) gives a theoretical discussion from which estimates of P_0 and R' may be obtained. Smith, Archambeau & Gile (1969) show that an estimate of P_0 and R' may also be obtained from observations of the near source strain change accompanying an explosion. It is worth noting that R' is not in general equal to either of the previous source radii used, namely R_c or R_0, but it is observed to be close to R_0 (see Smith *et al.*).

The inference of the pressure function $p_0(t)$ by Toksöz *et al.* is based on the far field seismic observations and these particular data are necessarily quite severely band limited due to observational difficulties and uncertainties related to path effects. Hence the frequency dependence of the effective pressure function, while consistent with the limited seismic observations, may not represent an appropriate dependence at all frequencies. An alternative approach is to use close-in observations or numerical shock wave code calculations appropriate to the region at or near the elastic zone boundary to obtain a description of the effective seismic source for the explosion itself. One approach is to express the close-in elastic field in terms of a reduced potential $\psi(t-r/v_p)$, valid in the elastic zone. This potential is defined by

$$u_r = -\frac{\partial}{\partial r}(\psi(t-r/v_p)/r). \tag{5.18}$$

The potential is seen to be the ordinary scalar potential with $\tau = t-r/v_p$ the reduced or retarded time. In terms of the radial displacement due to the explosion, where it is generally assumed for close-in measurement work that u_r represents only the effect of the explosion itself,

$$\psi(\tau) = rv_p \exp(-v_p \tau/r) \int_0^\infty u_r(\xi) \exp(v_p \xi/r) \tag{5.19}$$

where this is obtained directly from (5.18). Here u_r is considered to be an observed quantity and ψ is computed from those observations. In this discussion it is useful to explicitly note that ψ is defined at the elastic zone radius R' and so it will be denoted as $\psi(\tau, R')$.

For distances $r > R'$, we can express the transform of the radial displacement, $\tilde{u}_r(\mathbf{r}, \omega)$, in terms of the transform of $\psi(\tau, R')$, as

$$\tilde{u}_r(\mathbf{r}, \omega) = \Psi(R', \omega)\{1+ik_p r\}\frac{\exp(-ik_p r)}{r^2} \tag{5.20}$$

by simply taking the Fourier transform of (5.18). Here

$$\Psi(R', \omega) = \int_{R'/v_R}^\infty \psi(t, R')\exp(-i\omega t)\,dt = \exp(-ik_R R')\int_0^\infty \psi(\tau, R')\exp(-i\omega\tau)\,d\tau$$

is the transform of ψ, which is one sided since $\psi(\tau, R') = 0$ for $\tau < 0$. Comparing this with (5.15a) in $|\mathbf{r}| > R'$ we see that $\Psi(R', \omega)$ can be related to the equivalent pressure transform, $\tilde{p}_0(\omega)$, through the identification of the coefficients of the r dependent factor common to both expressions. Thus since we again define $t = 0$ as the initiation of the explosion and R'/v_R the time at which the elastic radius is defined, we have

$$\Psi(R', \omega) = \frac{\tilde{p}_0(\omega)}{4\mu} R'^3 \left[\frac{\exp[-i(\theta_p - k_p R')]}{\left[\left(1-\frac{k_s^2 R'^2}{4}\right)^2 + (k_p^2 R'^2)\right]^{\frac{1}{2}}}\right]. \tag{5.21}$$

Hence, with $\Psi(R', \omega)$ specified at the radius R' we can determine the appropriate 'cavity pressure' $\tilde{p}_0(\omega)$ to be used in the far field theory and for comparisons with pressure functions inferred from far field observations. More simply, the specified form of ψ obtained from observations or from numerical shock wave codes appropriate to general media and different explosive yields, can be used directly to predict the far field radiation. Thus empirical data for ψ, such as those given by Haskell (1967), can be used to predict the seismic spectrum in realistic Earth models. Adding the explosive field to that due to the tectonic effects discussed can therefore yield a prediction of the entire seismic field in quite general circumstances.

Summary

It appears that, at least in some cases, all the anomalous radiation from explosions can be explained by the stress relaxation accompanying the creation of the (roughly spherical) shatter zone in a prestressed medium. In other cases prestress levels, weak zones, or faulting within the material as well as dynamical or static overloading may be such that secondary rupture with distinct fault symmetry may also occur and the resulting stress relaxation due to creation of such a rupture also contributes to the anomalous radiation field.

In this paper we have confined our discussion to the theoretical aspects of stress wave radiation due to the spherical rupture zone created by the explosive shock wave and cavity overpressure in a prestressed medium. In this regard we have derived the total energy radiated and the spectrum and spatial pattern of the complete radiation field for several models of rupture formation. We find that the spatial pattern of the anomalous field is equivalent to a double couple equivalent source and this is what has been observed. We also note that the expressions for the radiation field allow us to determine source parameters by fitting the theory to observations, and Archambeau & Sammis (1970) have done this. They find reasonable values for the rupture radius R_0 (420 m) and initial prestress field (75 b) in the case of the Bilby explosion ($m_b = 5{\cdot}8$).

It remains to apply these theoretical predictions to other explosions for which we can be certain that no appreciable secondary faulting occurs, which if present would invalidate the assumption of nearly spherical symmetry made in this work. Extensions of this basic theory to earthquake geometries and rupture rates is clearly called for in the case of secondary rupturing and of course in order to treat natural earthquakes as well. Archambeau (1964, 1968) has considered such cases and use of the theoretical results should provide interesting tectonic information and an insight into processes involved in earthquake mechanics. The basic theoretical results given in this paper do, however, in large part apply directly to the more complex earthquake geometries. In addition, for very deep earthquakes in trenches or for volcanic types of earthquakes, the models adopted here may be quite good approximations inasmuch as the spherical symmetry and boundary conditions assumed should again be appropriate. Thus these models could have a wider application.

The principal results of this study are

(1) Theoretical relations for the seismic energy radiated due to stress relaxation around a pressurized fluid-like failure zone are obtained as a function of the radial dimension of the relaxation zone.

(2) An estimate of the partition of the tectonic energy between compressional waves ($E_R^{(p)}$) and shear waves ($E_R^{(s)}$) shows that $E_R^{(p)}/E_R^{(s)} \simeq \frac{1}{10}$ for explosive induced tectonic effects.

(3) The tectonic energy release due to explosive induced rupture is equal to the energy change due to creation of a dislocation on the rupture surface of an amount $\Delta u_i = u_i^*(\Sigma^-)$, where $u_i^*(\Sigma^-)$ is the change in the displacement field on the outside of the failure surface Σ, and where the dislocation must give rise to a stress field which is the mean of the initial and final stress fields.

(4) The theoretical expression of the dynamical field in terms of a generalized Green's function representation with several rupture formation models is evaluated in detail. The tectonic field is quadrupole in form and the radiation pattern shape (or symmetry) is not dependent on frequency, remaining purely quadrapole at all frequencies for all models. Low frequency limits are evaluated in detail. The field due to the compressional effects of the explosion itself is given to complete the description of the total field.

In addition to the purely analytical results given in this paper it is important to investigate the consequences of these predictions in a systematic fashion. This amounts to a computational program which is most usefully coupled with comparisons to observations of spectra, radiation patterns and the like for particular types of seismic waves. Such an analysis is therefore most reasonably carried out in a separate study.

Acknowledgments

The author is grateful to G. Frazier, T. Hanks, A. Linde and J. Minster for discussions of this work which led to some useful clarifications.

This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract No. F44620-69-C-0067.

Seismological Laboratory,
California Institute of Technology
Pasadena,
California 91109.

References

Aki, K., 1964. A note on surface wave generation from the Hardhat nuclear explosion, *J. geophys. Res.*, **69,** 1131–1134.

Aki, K., Reasenberg, P., DeFazio, T. & Tsai, Y. B., 1969. Near-field and far-field seismic evidence for triggering of an earthquake by the Benham explosion, *Bull. seism. Soc. Am.*, **59,** 6, 2197–2207.

Archambeau, C. B., 1964. *Elastodynamic source theory*, Ph.D. thesis, California Institute of Technology, Pasadena.

Archambeau, C. B., 1968. General theory of elastodynamic source fields, *Rev. Geophys.*, **16,** 241–288.

Archambeau, C. B. & Minster, J. B., 1972. Representation theorems in prestressed media with moving boundaries, in preparation.

Archambeau, C. B., Flinn, E. A. & Lambert, D. G., 1966. Detection, analysis and interpretation of teleseismic signals, 1, *J. geophys. Res.*, **71,** 3483–3501.

Archambeau, C. B., Flinn, E. A. & Lambert, D. G., 1969. Fine structure of the upper mantle, *J. geophys. Res.*, **74,** 25, 5825–5865.

Archambeau, C. B. & Sammis, C., 1970. Seismic radiation from explosions in prestressed media and the measurement of tectonic stress in the earth, *Rev. Geophys.*, **8,** 473–499.

Bishop, R. H., 1963. Spherical shock waves from underground explosions, close-in phenomenon of buried explosions, Final Rept. SC–4907(RR), pp. 115–158, Scandia Corporation.

Brune, J. N. & Pomeroy, P. W.. 1963. Surface wave radiation patterns for underground nuclear explosions and small magnitude earthquakes, *J. geophys. Res.*, **68,** 5005–5028.

Chinnery, M. A., 1964. The strength of the Earth's crust under horizontal shear stress, *J. geophys. Res.*, **69,** 2085–2089.

Erdelyi, A., Magmus, W., Oberhettinger, F. & Tricomi, F. G., 1954. *Tables of Integral Transforms, vol.* 1–2, *Bateman Manuscript Project, California Institute of Technology*, McGraw–Hill Book Company, New York.

Eshelby, J. D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems, *Proc. R. Soc.*, **A241,** 376–396.

Harkrider, D. G., 1963. *Rayleigh and Love waves from sources in a multilayered elastic half-space, Part II*, Thesis, California Institute of Technology.

Haskell, N. A., 1961. A static theory of the seismic coupling of a contained underground explosion, *J. geophys. Res.*, **66,** 2937–2944.

Haskell, N. A., 1967. Analytic approximation for the elastic radiation from a contained underground explosion, *J. geophys. Res.*, **72,** 2583–2587.

Kaula, W. M., 1963. Elastic models of the mantle corresponding to variations in the external gravity field, *J. geophys. Res.*, **68,** 4967–4978.

Knopoff. L., 1958. Energy release in earthquakes, *Geophys. J. R. astr. Soc.*, **1,** 44–52.

Lambert, D. G., Flinn, E. A. & Archambeau, C. B., 1971. Seismic radiation and source parameters for earthquakes and underground explosions, submitted to *J. geophys. Res.*

Landau, L. D. & Lifshitz, E. M., 1959. *Theory of Elasticity*, Pergamon Press Ltd, Oxford.

Morse, P. M. & Feshbach, H., 1953. *Methods of Theoretical Physics*, McGraw–Hill.

Press, F. & Archambeau, C., 1962. Release of tectonic strain by underground nuclear explosions, *J. geophys. Res.*, **67,** 337–343.

Sharpe, J. A., 1942. The production of elastic waves by explosive pressures: I, Theory and empirical field observations, *Geophysics*, **7,** 144–154.

Smith, S. W., Archambeau, C. B. & Gile, W., 1969. Transient and residual strains from large underground explosions, *Bull. seism. Soc. Am.*, **56,** 2185–2196.

Sokolnikoff, I. S., 1956. *Mathematical Theory of Elasticity*, McGraw–Hill Book Co., New York.

Steketee, J. A., 1958. Some geophysical applications of the elasticity theory of dislocations, *Can. J. Phys.*, **36,** 1168–1198.

Toksöz, M. N., Ben-Menahem, A. & Harkrider, D. G., 1964. Determination of source parameters of explosions and earthquakes by amplitude equalization of surface waves: Pt. 1, Underground nuclear explosions, *J. geophys. Res.*, **69,** 4355–4366.

Toksöz, M. N., Harkrider, D. G. & Ben-Menahem, A., 1965. Determination of source parameters by amplitude equalization of seismic surface waves: 2, Release of tectonic strain by underground nuclear explosions and mechanisms of earthquakes, *J. geophys. Res.*, **70,** 907–922.

Wyss, M., 1970. Stress estimates for South American shallow and deep earthquakes, *J. geophys. Res.*, **75,** 1529–1544.